Chern-Weil theorem, Lovelock Lagrangians in critical dimensions and boundary terms in gravity actions

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In this paper we show how to translate into tensorial language the Chern-Weil theorem for the Lorentz symmetry, which equates the difference of the Euler densities of two manifolds to the exterior derivative of a transgression form. For doing so we need to introduce an auxiliary, hybrid, manifold whose geometry we construct explicitly. This allows us to find the vector density, constructed out of spacetime quantities only, whose divergence is the exterior derivative of the transgression form. As a consequence we can show how the Einstein-Hilbert, Gauss-Bonnet and, in general, the Euler scalar densities can be written as the divergences of genuine vector densities in the critical dimensions \(D = 2, 4\), etc. As Lovelock gravity is a dimensional continuation of Euler densities, these results are of relevance for Gauss-Bonnet and, in general, Lovelock gravity. Indeed, these vectors which can be called generalized Katz vectors ensure, in particular, a well-posed Dirichlet variational principle.

I. INTRODUCTION

It is well known that the Einstein tensor is identically zero in two dimensions and that the Gauss-Bonnet tensor is identically zero in four dimensions. The easiest way to show this fact in tensorial language is to write these tensors \(\hat{\text{la}}\) Lovelock \(^1\) using the generalized Kronecker symbol (see also \(\hat{\text{la}}\) \(^2\)). These tensors being, up to specific divergences, the variational derivatives of the Einstein-Hilbert (EH) or Gauss-Bonnet (GB) Lagrangians, a number of authors \(^3\), see also \(\hat{\text{la}}\) \(^4\)-\(^7\), have stated that the Lagrangians themselves could be written (in the critical dimensions two or four) as divergences of some objects, since the variational derivative of a divergence is identically zero.

Now, since the EH and the GB Lagrangians are scalar densities containing second derivatives of the metric at most, they must be divergences of vector densities containing at most first derivatives of the metric. However, it is impossible to build a vector density out of the metric and its derivatives alone. Therefore, another ingredient must be added. For example, in his proof \(^8\) that the Lovelock scalar densities are indeed the divergences of true to gods vector densities \(V^\mu\), Horndeski had to introduce an arbitrary non-null contravariant vector \(U^\mu\).

One can also follow the formalism of Myers \(^9\) to show that the Einstein and Gauss-Bonnet tensors are identically zero in the critical dimensions by relating the corresponding EH and GB actions directly to surface terms, without trying first to write them as the divergences of vector densities. However, it turns out that Myers surface terms are in fact the radial components of vector densities (something which is not guaranteed \(\hat{\text{a priori}}\) for any boundary term). Indeed, as we show explicitly in Appendix \(\hat{\text{A}}\) the radial components of Horndeski’s \(V^\mu\) reproduce Myers’ boundary terms in the critical dimensions, when the extra vector \(U^\mu\) is chosen to be the normal to the boundary.

Now, whereas Horndeski’s proof is purely tensorial and introduces explicitly an extra vector, Myers uses the vielbein language where the invariance under diffeomorphisms and the Lorentz symmetry are restricted to the boundary, a fact which, as we will see below, hides the necessity of introducing an extra structure.

Our approach to show that the Lovelock scalar densities can be written as the divergences of explicit vector densities in the critical dimensions will rely on the translation of the Chern-Weil (CW) theorem (see, e.g., \(^11\)) for the Lorentz symmetry, which is at the heart of Myers’ proof, into fully covariant spacetime tensorial language. The CW theorem states that in \(D = 2p\) dimensions the difference of the Euler densities of two manifolds is equal to the exterior derivative of a \(2p - 1\)-form, which is called a transgression form (TF). Since this theorem involves two different manifolds, the needed extra structure, instead of the extra vector introduced by Horndeski, will be one of the two manifolds, that we will refer to as the background.

This translation is interesting for a number of reasons. First, it confirms that relating the Lorentz gauge invariance of transgressions forms and the invariance under general diffeomorphisms of boundary terms in gravity theories requires the introduction of an additional structure. Second, the divergences of the vector densities we shall construct, which can rightly be called generalized Katz vector densities \(^12\), when added to the dimensionally continued Lovelock actions, guarantee that their variations with respect to the metric obey Dirichlet boundary conditions. These Katz vectors also ensure, with a proper choice of the background manifold, that the actions are finite on shell as well as the corresponding Noether charges. Indeed, it was shown in Ref. \(^13\) (see also \(^14\)), for the EGB gravity case,
that adding the divergence of the generalized Katz vector density to the action provides simultaneously the correct conserved charges together with a well-defined variational principle. However, not much detail was given there about the geometrical meaning of its construction. Thus, the present work is also intended to fill this gap.

More precisely, we will show that the generalized Katz vector densities, that we shall construct with geometrical objects associated with two manifolds $\mathcal{M}$ and $\mathcal{N}$, are directly related with a transgression form constructed with the spin connections associated with $\mathcal{M}$ and an auxiliary, hybrid, manifold $\mathcal{N}$, whose geometry we shall completely characterize.

As a consequence, we will show that if the background is chosen in such a way that the Euler density of the associated hybrid manifold vanishes, then the Einstein-Hilbert, Gauss-Bonnet and, in general, the Lovelock Lagrangians reduce to the divergence of a vector density constructed with spacetime tensors in the critical dimensions $D = 2, 4, \text{ etc.}$ Moreover, using Gauss coordinates for a radial foliation, the radial component of this vector reproduces Myers' boundary terms. This shows explicitly that, to relate Myers' terms with the divergence of vector densities constructed with spacetime quantities, an extra structure is indeed required.

This article is organized as follows. In Section II we give the main ingredients we will use, namely a brief review about the vielbein formalism and the Chern-Weil theorem. In Section III we explain why in general it is not possible to make a full translation of a Lorentz transgression form to tensorial language and analyze the differences between Lorentz and spacetime tensors with respect to two different manifolds. Then, in Section IV we introduce the hybrid manifold that allows us to obtain the tensorial version of the Chern-Weil theorem. Finally, Section V contains some further comments.

II. PRELIMINARIES

The text-book material presented in this preliminary section is due to fix our conventions and notations.

A. Vielbein formalism: a recap

The vielbein $e^A = e^A dx^\mu$ and spin connection $\omega^A_B = \omega^A_B dx^\mu$, where $x^\mu = t, r, \phi_1, \ldots, \phi_{D - 2}$ are spacetime coordinates and $A, B = 0, \ldots, D - 1$ are Lorentz indices, are one-forms allowing to describe the geometry of a $D$-dimensional manifold $\mathcal{M}_D$ in a way similar to what is done in the tensorial language by means of the metric and the affine connection $(g_{\mu\nu}, \Gamma_{\mu\nu}^\alpha)$. The main difference is that the vielbein formulation makes explicit reference to the local Lorentz symmetry as an internal gauge symmetry. Denoting by $e^A_\mu$ the inverse matrix of the vielbein components $e^A_\mu$, such that $e^A_\mu e^\mu_A = \delta^\mu_B$ and $e^A_\mu e^\mu_B = \delta^B_A$, the relation between both languages is given by,

$$\eta_{AB} = e^A_\mu e^\mu_B g_{\mu\nu},$$ (2.1)

$$\omega^A_{\mu B} = e^A_\mu \epsilon^B_{\gamma} \Gamma^\alpha_{\mu\gamma} + e^A_\alpha \partial_\mu e^\alpha_B.$$ (2.2)

The first relation (2.1) states that in each point of $\mathcal{M}_D$ it is possible to find an invertible coordinate transformation $x^\mu = x^\mu(y^A)$ such that the Jacobian matrix $e^A_\mu = \partial x^\mu/\partial y^A$ brings $g_{\mu\nu}$ to the Minkowski metric $\eta_{AB}$. Thus, the vielbein components $e^A_\mu$ is the Jacobian of the inverse transformation $e^A_\mu = \partial y^A/\partial x^\mu$. By construction, the vielbein $e^A$ and spin connection $\omega^A_B$ are invariant under coordinate transformations $x'^\mu = x^\mu(x^\nu)$, while under a local Lorentz transformation $y^A = \Lambda^A_B y^B$ (with $\Lambda^T \eta \Lambda = \eta$ and $\eta$ being the Lorentz metric) they transform respectively as

$$e^A = \Lambda^A_B e^B,$$ (2.3)

$$\omega^A_{\mu B} = \Lambda^A_C \Lambda^D_B \omega^D_{\mu C} + \Lambda^A_C d\Lambda^C_B,$$ (2.4)

where $\Lambda^A_B$ denotes the inverse of $\Lambda^B_A$ and $d$ is the exterior derivative. In particular, given a metric tensor $g_{\mu\nu}$, the vielbeines can be determined up to a Lorentz transformation and therefore $e^A_\mu$ carries the same number of independent components as $g_{\mu\nu}$.

The second relation (2.2), also known as the tetrad postulate, implies that the curvature and torsion two-forms defined as

$$\Omega^A_B = d\omega^A_B + \omega^A_C \omega^C_B = \frac{1}{2} \Omega^A_{B\mu} dx^\mu dx^\nu,$$ (2.5)

$$T^A = De^A = \frac{1}{2} T^A_{\mu\nu} dx^\mu dx^\nu,$$ (2.6)

$$\text{with} \quad De^A = \frac{1}{2} T^A_{\mu\nu} dx^\mu dx^\nu,$$
where $D$ defines the Lorentz covariant derivative, are related with the Riemann and torsion tensors $R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \ldots$ and $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$ by

$$\Omega^\lambda_{\beta\mu\nu} = \epsilon_\lambda^{\alpha\beta} R^\alpha_{\beta\mu\nu} \quad \text{and} \quad T^\lambda_{\mu\nu} = \epsilon_\lambda^\alpha T^\alpha_{\mu\nu}. \quad (2.7)$$

With this notation the Ricci scalar is given by $R = g^{\mu\nu} R_{\mu\nu}$ with $R_{\mu\nu} = R^\alpha_{\mu\nu\alpha}$ being the Ricci tensor. Remark that, for shortness, we omit the wedge product between differential forms.

The manifold $\mathcal{M}_D$ is pseudo-Riemannian if it satisfies the metricty condition $\nabla_\mu g_{\nu\rho} = 0$ (here $\nabla$ stands for covariant derivative with respect to $\Gamma$) and the torsionless condition $T^\lambda_{\mu\nu} = 0$. The only connection satisfying simultaneously these conditions is the Christoffel connection, which is completely determined by the metric and its derivatives, $\Gamma^\alpha_{\nu\gamma} = \Gamma^\alpha_{\nu\gamma} (g, \partial g)$. Similarly, in the vielbein formalism, a pseudo-Riemannian geometry is characterized by the conditions $D_\mu A_{\nu} = 0$ and $T^A = 0$. The first one is equivalent to assume that the spin connection is antisymmetric (i.e., $\omega^{AB} = -\omega^{BA}$ with $\omega^{AB} = \eta^{BC} \omega^A_C$) and together with the torsionless condition one is able to solve the spin connection in terms of the vielbein and its derivatives, $\omega^A_B = \omega^A_B (e, \partial e)$, the explicit expression of which will not be needed here.

Finally, the Levi-Civita symbols $\epsilon_{\mu_1 \cdots \mu_D}$ and $\epsilon_{A_1 \cdots A_D}$ together with $\epsilon^{\mu_1 \cdots \mu_D}$ and $\epsilon^{A_1 \cdots A_D}$ are such that $\epsilon_{\mu_1 \cdots \mu_D} = -\epsilon_{\mu_1 \cdots \mu_D}$ and $\epsilon^{A_1 \cdots A_D} = -\epsilon^{A_1 \cdots A_D}$ with the convention $\epsilon^{r_1 \cdots r_{D-2}} = \epsilon^{012 \cdots D} = 1$. It is easy to show that under a coordinate tranformations $\epsilon_{\mu_1 \cdots \mu_D}$ and $\epsilon^{\mu_1 \cdots \mu_D}$ transform respectively as tensorial densities of weight 1 and $-1$. As for $\epsilon_{A_1 \cdots A_D}$ and $\epsilon^{A_1 \cdots A_D}$ they transform as tensors under local Lorentz transformations. Moreover, both are related by

$$\epsilon_{A_1 \cdots A_D} \epsilon^{A_1} \cdots \epsilon^{A_D} = \sqrt{-g} \epsilon_{\mu_1 \cdots \mu_D}, \quad (2.8)$$

where $\epsilon^A_\mu = \partial y^A / \partial x^\mu$.

### B. Chern-Weil theorem

The Chern-Weil theorem (see, e.g., [11]) was developed in quest for a proof of the generalized Gauss-Bonnet theorem. It is regarded a milestone towards a complete theory of characteristic classes which relates and unifies concepts in algebraic topology and differential geometry. It is formulated in terms of fiber bundle structures, a powerful tool that allows to build a gauge theory over a smooth manifold. Its basic ingredients are a Lie algebra with generators $T_M$, a Lie valued gauge connection one-form $A$ and its corresponding field strength $F = dA + A \wedge A$. It is easy to show that $(F^P)$, where $(\cdot)$ stands for the symmetrized trace of the generators, is invariant under gauge transformations in 2$p$-dimensions and thus, it is a topological term. The Chern-Weil theorem states that, given two connections $A$ and $A$, the topological terms constructed with their corresponding curvatures are closed forms and that the difference $(F^P) - (F^P)$ is an exact form, i.e., is the exterior derivative of an odd-form which is known as transgression form (see Appendix [12] for its general expression). In particular, a Chern-Simons form is recovered from a transgression form by setting the second connection to zero.

In the case where the symmetry is described by the Lorentz algebra, the Euler topological term for a 2$p$-dimensional pseudo-Riemannian manifold $\mathcal{M}_{2p}$ (with $p$ an integer) is defined in the vielbein formalism as

$$E_{2p} (\Omega) \equiv \epsilon_{A_1 \cdots A_{2p}} \Omega^{A_1 A_2} \cdots \Omega^{A_{2p-1} A_{2p}}, \quad (2.9)$$

where $\Omega^{AB} = \eta^{AC} \Omega^A_C$. This quantity is, by construction, a 2$p$-form invariant under local Lorentz transformations. It is a topological term because, as stated by the Gauss-Bonnet theorem, its integral over a compact manifold is related with the Euler characteristic $\chi (\mathcal{M}_{2p})$ which describes its topology. Further details can be found, e.g., in Ref. [13].

Consider now a second pseudo-Riemannian manifold $\mathcal{M}_{2p}$ with Lorentz connection $\bar{\omega}^A_B$, curvature $\bar{\Omega}^B_D = d\bar{\omega}^B_D + \bar{\omega}^A_C \bar{\omega}^C_D$. Using that by definition the Lorentz tensors $\eta_{AB}$ and $\epsilon_{A_1 \cdots A_{2p}}$ are the same for both manifolds (because the Minkowski tangent space is the same for each point of each manifold), we can define $\bar{\omega}^{AB} = \eta^{BC} \bar{\omega}^A_C$ and $\bar{\Omega}^{AB} = \eta^{BC} \bar{\Omega}^A_C$ so that the Euler term in $\mathcal{M}_{2p}$ is given by

$$E_{2p} (\bar{\Omega}) = \epsilon_{A_1 \cdots A_{2p}} \bar{\Omega}^{A_1 A_2} \cdots \bar{\Omega}^{A_{2p-1} A_{2p}}. \quad (2.10)$$

Now, the Chern-Weil (CW) theorem for the Lorentz symmetry establishes that the difference between the two topological terms (2.9) and (2.10) is an exact form, i.e., the exterior derivative of a (2$p$ - 1)-form $T^{(2p-1)}$, called transgression form, which is completely determined by the connections $\omega$ and $\bar{\omega}$:

$$E_{2p} (\Omega) - E_{2p} (\bar{\Omega}) = dT^{(2p-1)}. \quad (2.11)$$
For example, for \( p = 1 \) the antisymmetric property of the spin connections \( \omega \) and \( \bar{\omega} \) leads \( \Omega^{AB} = d\omega^{AB} \) and \( \bar{\Omega}^{AB} = d\bar{\omega}^{AB} \) and thus, the difference of the two Euler terms \( \mathcal{E}_2 (\Omega) = \varepsilon_{AB} \Omega^{AB} \) and \( \mathcal{E}_2 (\bar{\Omega}) = \varepsilon_{AB} \bar{\Omega}^{AB} \) is simply given by

\[
\mathcal{E}_2 (\Omega) - \mathcal{E}_2 (\bar{\Omega}) = d[\mathcal{T}^{(1)}(\bar{\theta})], \quad \text{with} \quad \mathcal{T}^{(1)}(\bar{\theta}) = \varepsilon_{AB} \bar{\theta}^{AB} \quad \text{and} \quad \bar{\theta}^{AB} \equiv \omega^{AB} - \bar{\omega}^{AB}.
\]  

(2.12)

This is the simplest realization of the Chern-Weil theorem for the Lorentz symmetry. As reviewed in the Appendix [B] for higher values of \( p \) the transgression form is given by

\[
\mathcal{T}^{(2p-1)}(\theta, \Omega, \bar{\Omega}) = p \int_0^1 dt \varepsilon_{A_1 \ldots A_{2p}} \bar{\theta}^{A_1 A_2} \Omega^{A_3 A_4} \ldots \Omega^{A_{2p-1} A_{2p}},
\]

(2.13)

where \( \Omega^{AB} = d\omega^{AB} + \omega^A (t) C^B(t) \) and \( \bar{\omega}^{AB} = \bar{\omega}^{AB} + t\bar{\theta}^{AB} \) is a connection which interpolates between \( \bar{\omega}^{AB} \) and \( \omega^{AB} \) for \( t \in [0,1] \). In Appendix [B] it is also shown that the interpolating curvature has the following alternative expressions,

\[
\Omega^{AB} = t\Omega^{AB} + (1 - t) \bar{\Omega}^{AB} - t(1 - t) \bar{\theta}^A_C \theta^{CB} \quad \text{(2.14)}
\]

\[
= \Omega^{AB} + t \bar{D} \bar{\theta}^{AB} + t^2 \bar{\theta}^A_C \theta^{CB} \quad \text{(2.15)}
\]

\[
= \Omega^{AB} + (t - 1) D \bar{\theta}^{AB} + (t - 1)^2 \bar{\theta}^A_C \theta^{CB}, \quad \text{(2.16)}
\]

Transgression forms (TFs) have proved to be useful to deal with a number of different physical situations. Originally used to treat the general problem of anomalies in field theory \([16\text{--}18\text{]}\), more recent applications range from the study of anomalies in hydrodynamics in the context of gauge/gravity duality \([19\text{]}\) to holographic models of baryons \([20\text{]}\).

In the context of gravity, the use of TFs is possible whenever the Lie algebra accounts for the symmetries of the Lagrangian (e.g., Lorentz, (anti-)de Sitter, etc.). In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories. In particular, TFs have been used in a dimensionally continued version to define a well-posed variational principle in different gravity theories: The Gibbons-Hawking-York (GHY) boundary term \([21,22\text{]}\), and its generalization by Myers \([9\text{]}\) to the case of Lovelock theories \([1\text{]\], that defines a Dirichlet version to define a well-posed variational principle in different gravity theories.

III. ISSUES ABOUT THE TENSORIAL TRANSLATION

A. The problem

As it has been reviewed in the previous section, given two pseudo-Riemannian manifolds \( M_{2p} \) and \( M_{2p} \) of dimension \( 2p \), the Chern-Weil theorem states that:

\[
\mathcal{E}_{2p} (\Omega) - \mathcal{E}_{2p} (\bar{\Omega}) = d\mathcal{T}^{(2p-1)},
\]

(3.1)

where \( \mathcal{E}_{2p} (\Omega) \), \( \mathcal{E}_{2p} (\bar{\Omega}) \) are the topological terms defined by (2.9,2.10) and where \( \mathcal{T}^{(2p-1)} \) is the transgression form defined by (2.11,13), which depends on the connections \( \omega \) and \( \bar{\omega} \) through the Lorentz covariant objects \( \Omega^{AB} \), \( \bar{\Omega}^{AB} \) and \( \bar{\theta}^{AB} \) (see Eqs. (2.11,2.13)). The l.h.s. of (3.1) can be translated rightaway into tensorial language as

\[
\mathcal{E}_{2p} (\Omega) - \mathcal{E}_{2p} (\bar{\Omega}) = \frac{1}{2^p} \delta^{\gamma_1 \ldots \gamma_{2p}} \left[ \sqrt{-g} R^{\alpha_1 \alpha_2} \ldots R^{\alpha_{2p-1} \alpha_{2p}} - \sqrt{-g} \bar{R}^{\alpha_1 \alpha_2} \ldots \bar{R}^{\alpha_{2p-1} \alpha_{2p}} \right] dx^\gamma, \quad (3.2)
\]

i.e., it can be written as an expression depending only on spacetime tensorial objects such as \( g_{\mu\nu} \), \( \bar{g}_{\mu\nu} \), \( R^\alpha_{\gamma\mu\nu} \) and \( \bar{R}^\alpha_{\gamma\mu\nu} \). The translation can be easily made using the relations

\[
\Omega^{AB} = \frac{1}{2} \varepsilon^A_C e^B_D \Gamma^\alpha_{\mu\nu} dx^\mu dx^\nu, \quad \bar{\Omega}^{AB} = \frac{1}{2} \varepsilon^A_C e^B_D \bar{\Gamma}^\alpha_{\mu\nu} dx^\mu dx^\nu,
\]

\[
R^\alpha_{\gamma\mu\nu} = g^{\bar{\beta} \bar{\gamma}} \bar{R}^\alpha_{\bar{\gamma}\mu\nu}, \quad \bar{R}^\alpha_{\gamma\mu\nu} = \bar{g}^{\bar{\beta} \bar{\gamma}} \bar{R}^\alpha_{\bar{\gamma}\mu\nu},
\]

(3.3)
which hold due to the tetrad postulates $\omega^{AB}_\mu = e^A_\alpha e^{B\gamma}_\mu \Gamma^{\alpha\gamma}_\mu + e^A_\alpha \partial_\mu e^{B\alpha}$ and $\bar{\omega}^{AB}_\mu = \bar{e}^A_\alpha e^{B\gamma}_\mu \Gamma^{\alpha\gamma}_\mu + \bar{e}^A_\alpha \partial_\mu \bar{e}^{B\alpha}$, together with the identities

$$\varepsilon A_1 \ldots A_p e^{\gamma_1}_{\mu_1} \ldots e^{\gamma_p}_{\mu_p} = \sqrt{-g} \varepsilon \mu_1 \ldots \mu_{2p}$$

$$\varepsilon A_1 \ldots A_p \bar{e}^{\gamma_1}_{\mu_1} \ldots \bar{e}^{\gamma_p}_{\mu_p} = \sqrt{-\bar{g}} \varepsilon \mu_1 \ldots \mu_{2p}$$

$$\varepsilon \mu_1 \ldots \mu_{2p} \varepsilon^{\gamma_1}_{\mu_1} \ldots \varepsilon^{\gamma_p}_{\mu_p} = -\delta^{\mu_1 \ldots \mu_{2p}}_{\mu_1 \ldots \mu_{2p}}, \quad dx^{\mu_1} \ldots dx^{\mu_{2p}} = -e^{\mu_1 \ldots \mu_{2p}} d^p x,$$  (3.4)

with $d^p x = dt d\theta_1 \ldots d\theta_{2p-2}$ and $\delta^{\mu_1 \ldots \mu_{2p}}_{\mu_1 \ldots \mu_{2p}}$ being the generalized Kronecker delta defined as the determinant of the $2p \times 2p$ matrix $\delta_{\mu_1 \ldots \mu_{2p}}^{\nu_1 \ldots \nu_{2p}}$ with $r, s = 1, \ldots, 2p$.

On the other hand, a complete translation in terms of spacetime tensors only of the r.h.s of (3.1) is not possible working with $M$ and $\bar{M}$ only. Indeed, the only thing that can be shown is that

$$d[T^{(2p-1)}(\bar{\theta}, \Omega, \bar{\Omega})] = \partial_\mu [\nu^\mu (\bar{\theta}, \Omega, \bar{\Omega})] d^p x,$$

where $\nu^\mu$ is a spacetime vector density which depends on $\Gamma, \bar{\Gamma}$ but also on $\epsilon, \bar{\epsilon}$: $\nu^\mu = \nu^\mu (\epsilon, \Gamma, \bar{\epsilon}, \bar{\Gamma})$ and, in general, there is no way to get rid of the vielbeins.

For example, in $D = 2$, the l.h.s. of Eq. (2.12) is given by $(\sqrt{-g} - \sqrt{-\bar{g}}) d^2 x$, while the r.h.s. is given by $\partial_\mu \nu^\mu d^2 x$ where $\nu^\mu$ is the following vector density

$$\nu^\mu = -\varepsilon_{AB} \epsilon^{\mu \nu} \bar{\theta}^A_{\nu} = -\varepsilon_{AB} \epsilon^{\mu \nu} (\omega^A_{\mu} - \bar{\omega}^A_{\mu}) = -\varepsilon_{AB} \epsilon^{\mu \nu} \left[ (e^A_\alpha e^{B\gamma}_\mu \Gamma^{\alpha\gamma}_\mu - e^A_\alpha \partial_\mu e^{B\alpha}) + (\bar{e}^A_\alpha \partial_\mu \bar{e}^{B\alpha} - e^A_\alpha \partial_\mu \bar{e}^{B\alpha}) \right],$$  (3.5)

where $\bar{\theta}^A_{\nu} = \omega^A_{\nu} - \bar{\omega}^A_{\nu}$ are the components of the one-form $\bar{\theta}^A$ and where we have used the tetrad postulate in the last equality. Thus $\nu^\mu = \nu^\mu (\epsilon, \Gamma, \bar{\epsilon}, \bar{\Gamma})$ and there is at first sight no way to completely translate $d[T^{(1)}(\bar{\theta})]$ to tensorial language. One may orient the corresponding vielbeins such that the $e^{-1} \partial e$ (inhomogeneous) terms vanish. However, even in that case, the resulting expression cannot be written in terms of $\Delta^\alpha_{\mu\nu}$, defined as

$$\Delta^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu}.$$  (3.6)

Indeed the factor $(ce\Gamma - \bar{e}\bar{\Gamma})$, which then captures the structure of $\nu^\mu$, cannot be transformed into an expression of the type $ce(\Gamma - \bar{\Gamma})$ which would transform it in a spacetime tensor density because, as we show in the next subsection, $c$ and $\bar{c}$ are different and hence cannot be related by a local Lorentz rotation.

### B. Lorentz versus spacetime tensors

Consider a pair of $D$-dimensional pseudo-Riemannian manifolds $\{ M, \bar{M} \}$ endowed respectively with the metrics $\{ g_{\mu\nu}, \bar{g}_{\mu\nu} \}$ and where $\{ \Gamma^\alpha_{\mu\nu}, \bar{\Gamma}^\alpha_{\mu\nu} \}$ are the corresponding Christoffel symbols. Also choose a mapping $\sigma$ between these manifolds allowing us to use the same coordinates $x^\mu$ for each point $P \in M$ and $\bar{P} = \sigma (P) \in \bar{M}$. This choice is always possible and, as a consequence, a coordinate transformation $x'^\mu = x'^\mu (x)$ in $M$ induces the same coordinate transformation in $\bar{M}$. This means that if $P^\mu_{\nu_1 \ldots \nu_p}$ and $\bar{P}^\mu_{\nu_1 \ldots \nu_p}$ are tensors defined respectively on $M$ and $\bar{M}$, then their linear combinations $a P^\mu_{\nu_1 \ldots \nu_p} + b \bar{P}^\mu_{\nu_1 \ldots \nu_p}$ are also true tensors, because the Jacobian matrices are the same.

$$a P^\mu_{\nu_1 \ldots \nu_p} (x') + b \bar{P}^\mu_{\nu_1 \ldots \nu_p} (x') = \frac{\partial x'^\mu_{\nu_1}}{\partial x^{\mu_{\nu_1}}} \ldots \frac{\partial x'^\mu_{\nu_p}}{\partial x^{\mu_{\nu_p}}} \left( a P^\nu_{\beta_1 \ldots \beta_q} (x) + b \bar{P}^\nu_{\beta_1 \ldots \beta_q} (x) \right).$$  (3.7)

Thus, we can deal with linear combinations of tensors, defined on different manifolds, that are simultaneously spacetime tensors on both manifolds. Among those tensors we have, e.g.,

$$Y^\mu_{\nu} = ag_{\mu\nu} + b\bar{g}_{\mu\nu}, \quad a, b \text{ being arbitrary constants},$$

$$\Delta^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \bar{\Gamma}^\alpha_{\mu\nu}.$$  (3.8)

as well as derived quantities, such as $\Delta^\alpha_{\mu\beta} = g^{\beta\nu} \Delta^\alpha_{\mu\nu}$ and $\nabla_\mu \Delta^\nu_{\beta\beta}$.

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1 The quantity $\nu^\mu$ is indeed a vector density because, for spacetime transformations, $\bar{\theta}^A_{\nu}$ is a covariant vector while $\varepsilon^{\mu \nu}$ is a contravariant tensor density. It is also clear that for Lorentz transformations $\nu^\mu$ behaves as an invariant because $\varepsilon_{AB}$ is a Lorentz covariant tensor or rank 2, while $\delta^{\mu \nu}$ is a contravariant tensor of rank 2.

2 However, as shown in Appendix, a version of the Chern-Weil theorem which is free of vielbeins can be formulated in the particular case $D = 2$ using the fact that in this dimension all the metrics are conformally equivalent.
A similar analysis can be done in the vielbein formulation, where the analogs of (3.8) and (3.9) are
\[ E^A = a e^A + b \bar{e}^A, \quad a, b \text{ being arbitrary constants}, \]
\[ \tilde{\theta}^A_B = \omega^A_B - \bar{\omega}^A_B. \]  
(3.10)

We recall that by definition the Lorentz tensors \( \eta_{AB} \) and \( \varepsilon_{A_1\ldots A_D} \) are the same for both manifolds and thus we have, for example,
\[ \omega^{AB} = \eta^{BC} \omega^A_C, \quad \Omega^{AB} = \eta^{BC} \Omega^A_C, \quad \bar{\omega}^{AB} = \eta^{BC} \bar{\omega}^A_C, \quad \bar{\Omega}^{AB} = \eta^{BC} \bar{\Omega}^A_C. \]
(3.11)

In particular, we recognize that the one-form \( \Theta^{AB} = \eta^{BC} \bar{\theta}^A_C \) is the one appearing as a fundamental object in the definition of the transgression form \( T^{(2p-1)} (\theta, \Omega) \) given in (2.13).

Now, the question is to determine in which cases a local Lorentz-invariant quantity, constructed with the Lorentz-covariant objects
\[ \{ \eta_{AB}, \varepsilon_{A_1\ldots A_D}, e^A, \bar{e}^A, E^A, \Omega^{AB}, \bar{\Omega}^{AB}, \tilde{\theta}^A_B \}, \]
(3.12)

knowing that the following identities hold (see Eqs. (2.2) and (2.8))
\[ \varepsilon_{A_1\ldots A_D} e^{A_1} \cdots e^{A_D} = \sqrt{-\eta} \varepsilon_{\mu_1\ldots \mu_D}, \]
(3.13)
\[ \varepsilon_{A_1\ldots A_D} \bar{e}^{A_1} \cdots \bar{e}^{A_D} = \sqrt{-\eta} \bar{\varepsilon}_{\mu_1\ldots \mu_D}, \]
(3.14)
\[ \bar{\omega}^\mu_{\mu} = e^A B e^\mu \Gamma_{\mu\gamma} + e^A B e^B e^\mu, \quad \bar{\omega}^\mu_{\mu} = e^A B e^\mu \Gamma_{\mu\gamma} - e^A B e^\mu, \]
(3.15)

and that, as shown in Appendix [1], the vielbeins are related by
\[ e^A = K^A_B e^B, \]
(3.16)

where \( K = K^A_B (x) \) is not a Lorentz rotation (i.e., \( K^T \eta K \neq \eta \)) because \( e \) and \( \bar{e} \) are inequivalent as they describe different geometries.

The Euler terms \( \mathcal{E}_2 \) (\( \Omega \)) and \( \mathcal{E}_2 \) (\( \tilde{\omega} \)) in Section IIIA are examples where a complete translation is possible. They depend only on \( \varepsilon_{A_1\ldots A_D} \) and the curvatures \( \Omega^{AB} \) and \( \bar{\Omega}^{AB} \). Thus, the use of (3.13),(3.15) allows us to translate these Euler terms in tensorial expressions free of Lorentz indices. On the other hand, in the same section, we have found problems to translate in tensorial language the exterior derivative of the transgression form \( T^{(2p-1)} (\omega, \bar{\omega}) \) defined in (2.13), which depends on the object (written here, for visual simplicity, in the case the vielbeins are properly oriented)
\[ \bar{\theta}^{AB} = \eta^{BC} \bar{\theta}^A_C = [e^A e^B \Gamma^\alpha_{\mu\gamma} \gamma^\alpha_k - e^A e^B \Gamma^\alpha_{\mu\gamma}] dx^\mu, \]
(3.17)

A factorization of a same type of vielbeins from \( \Theta^{AB} \), which depends on the object \( K^A_B (x) \) and \( e \), and \( \bar{e} \), cannot factorize the vielbeins \( e \). Thus, one must discard the use of the object \( \Theta^{AB} \) in the Lorentz covariant constructions that can be written in tensorial way.

From the previous analysis, it is therefore clear that working only with the manifolds \( \mathcal{M} \) and \( \bar{\mathcal{M}} \) is not sufficient to express the CW theorem in tensorial language. In the next section we will introduce a hybrid manifold which will do the job.

3 Indeed, using (2.2) and (2.4) is easy to show that under a Lorentz transformation \( E'^A = A^A_B E^B \) and \( \tilde{\theta}^A_B = A^A_C A^C_B \tilde{\theta}^C_D \).
4 As another example, consider the Lorentz invariant quantity \( E^A E^B \eta_{AB} \), where \( E^A \) is given in (3.8) and where a tensorial product is assumed between \( E^A \) and \( E^B \). Using (3.16) we obtain
\[ E^A E^B \eta_{AB} = (a^2 g_{\mu\nu} + ab (e^A e^D + \bar{e}^A \bar{e}^D)) \left( (K^{-1})^A_B \eta_{AB} + \delta^A_D e^C e^D \bar{g}_{CD} (x) \right) dx^\mu dx^\nu, \]
where we have also used the relation \( g_{CD} (x) = (K^{-1})^A_C (K^{-1})^B_D \eta_{AB} \) found in Eq. (12) of the Appendix [1]. We see that there is no direct way to write \( E^A E^B \eta_{AB} \) in terms of the tensorial objects (3.12), i.e., as an expression free of vielbeins. Thus, one should also discard the object \( E^A \) from the Lorentz covariant constructions that can be fully translated to tensorial language.
IV. A TENSORIAL CHERN-WEIL THEOREM

A. The hybrid manifold $\tilde{M}$

Let us define the one-form $\tilde{\omega}^\mu_B$ as

$$\tilde{\omega}^\mu_B = \tilde{\omega}^\mu_B dx^\mu \equiv \left( e_\alpha^A e_\beta^B \Gamma^\alpha_{\mu\beta} + e_\alpha^A \partial_{\mu} e_\beta^B \right) dx^\mu. \quad (4.1)$$

A direct calculation shows that the transformation law of this object under a local transformation $y^A = \Lambda^A_B y^B$ is the same as Eq. (2.13). Thus, $\tilde{\omega}$ is a spin connection allowing to define consistently the covariant derivative of any Lorentz tensor [4]. This connection has been recently introduced in Ref. [13] and it has been named hybrid connection because it depends on objects that belong to different spaces: the vielbein $e$ associated with the manifold $M$ and the Christoffel symbol $\bar{\Gamma}$ of the manifold $\tilde{M}$.

As a consequence, the difference between $\omega^{AB} = \eta^{BC} \omega^A_C$ and $\tilde{\omega}^{AB} = \eta^{BC} \tilde{\omega}^A_C$ is related with the tensorial object $\bar{\Gamma}^{\alpha}_{\mu\gamma} = \Gamma^{\alpha}_{\mu\gamma} - \Gamma^{\alpha}_{\mu\gamma}$ defined in (3.8) as follows

$$\tilde{\gamma}^{AB} \equiv \omega^{AB} - \tilde{\omega}^{AB} = e_\alpha^A e_\beta^B \bar{\Gamma}^{\alpha}_{\mu\beta} dx^\mu, \quad \text{with} \quad \bar{\Gamma}^{\alpha}_{\mu\beta} = \eta^{\beta\gamma} \Gamma^{\alpha}_{\mu\gamma}. \quad (4.2)$$

The fact that two vielbeins of the same type can be factorized from $\tilde{\gamma}^{AB}$ (just as it happens, e.g., for $\Omega^{AB} = \frac{1}{2} e_\alpha^A e_\beta^B R^{\beta\gamma}_{\mu\beta} dx^\mu dx^\nu$) is crucial to find a tensorial formulation of the Chern-Weil theorem. For example, in the two dimensional case, if we replace the general connection $\tilde{\omega}$ by $\tilde{\omega}$ in the r.h.s of Eq. (2.12) and use the identity (3.13) we get,

$$d[T^{(1)}(\tilde{\omega})] = d(\varepsilon_{AB} \tilde{\gamma}^{AB}) = d \left( \varepsilon_{AB} e_\alpha^A e_\beta^B \bar{\Gamma}^{\alpha}_{\mu\beta} dx^\mu \right) = \partial_{\mu} k^\mu_\beta dx^\beta, \quad (4.3)$$

where

$$k^\mu_\beta(1) = \sqrt{-g} \delta^{\alpha\beta}_\mu \bar{\Gamma}^{\alpha}_{\mu\gamma}. \quad (4.4)$$

is a vector density that depends on the tensorial quantities (3.12) only: It is the Katz vector density [12] in 2 dimensions.

Before showing how the l.h.s. of Eq. (2.12) would be modified if we change $\tilde{\omega}$ by $\tilde{\omega}$, a study of the geometric properties of the hybrid connection is needed.

As shown in the Appendix A, $\tilde{\omega}^{AB}$ is not antisymmetric, hence the associated manifold $\tilde{M}$ is not metric compatible. On the other hand, the antisymmetrized object $\tilde{\omega}^{[AB]} = \frac{1}{2}[\tilde{\omega}^{AB}]$ has all the required properties to define a Riemannian manifold $\tilde{M}$ (see Appendix B for demonstrations): it transforms as a Lorentz spin connection and is such that two vielbeins of the same type can be factorized from the difference between $\omega^{AB}$ and $\tilde{\omega}^{AB}$, that is

$$\tilde{\gamma}^{AB} \equiv \omega^{AB} - \tilde{\omega}^{AB} = e_\alpha^A e_\beta^B \bar{\Gamma}^{\alpha}_{\mu\beta} dx^\mu, \quad \text{with} \quad \bar{\Gamma}^{\alpha}_{\mu\beta} = \eta^{\beta\gamma} \Gamma^{\alpha}_{\mu\gamma}. \quad (4.5)$$

Therefore, we introduce the antisymmetric hybrid spin connection

$$\tilde{\omega}^{A}_{\mu B} \equiv \eta^{BC} \omega^{[AC]}_{\mu} = \frac{1}{2} \eta^{BC} \left( e_\alpha^A e_\tau^C \Gamma^\alpha_{\mu\tau} - e_\alpha^C e_\tau^A \Gamma^\alpha_{\mu\tau} + e_\alpha^A \partial_\mu e_\tau^C - e_\alpha^C \partial_\mu e_\tau^A \right), \quad (4.6)$$

which can be associated with an auxiliary manifold $\tilde{M}$ with metric $\tilde{g}_{\mu\nu}$, affine connection $\tilde{\Gamma}^{\alpha}_{\mu\gamma}$ and vielbein $e^A$ satisfying the basic relations

$$\tilde{g}_{\mu\nu} = \tilde{e}^A \tilde{e}^B \eta_{AB}, \quad (4.7)$$

$$\tilde{\omega}^{A}_{\mu B} = \tilde{e}^A \tilde{e}^D \tilde{\gamma}_{AB}^{D} + e_\alpha^A \partial_\mu e_\alpha^B. \quad (4.8)$$

Thus, Eq. (4.5) ensures that the Lorentz curvature and the torsion two-forms

$$\tilde{\Omega}^{A}_{B} = d\tilde{\omega}^{A}_{B} + \tilde{\omega}^{C}_{A} \tilde{\omega}^{B}_{C} = \frac{1}{2} \tilde{\Omega}^{A}_{B\mu \nu} dx^\mu dx^\nu, \quad \tilde{T}^{A} \equiv \tilde{D} e^A = \frac{1}{2} \tilde{T}^{A}_{\mu \nu} dx^\mu dx^\nu, \quad (4.8)$$

For example, if a Lorentz vector $V^A$ transforms as $V^A = \Lambda^A_B V^B$, then the covariant derivative $\tilde{D} V^A = dV^A + \tilde{\omega}^{A}_{\mu} V^B$ transforms as a vector too, i.e., $\tilde{D} V^A = \Lambda^A_B \tilde{D} V^B$. 

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are related with the Riemann and torsion tensors $R^\alpha_{\beta\mu\nu} = \partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_\nu \tilde{\Gamma}^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\rho} \tilde{\Gamma}^\rho_{\mu\nu} - \Gamma^\alpha_{\mu\rho} \tilde{\Gamma}^\rho_{\beta\nu}$ and $\tilde{\Gamma}^\mu_{\nu\rho} = \tilde{\Gamma}^\alpha_{\nu\rho} \tilde{\Gamma}^\mu_{\alpha\beta} - \tilde{\Gamma}^\alpha_{\nu\rho} \tilde{\Gamma}^\mu_{\beta\alpha}$.

We notice also that the Bianchi identities $\tilde{D} \tilde{\Omega}^A_B = 0$ and $\tilde{D} \tilde{\Omega}^A_B = \tilde{\Omega}^A_B$ are satisfied. Now, writing Eq. (4.0) as

$$\tilde{\omega}^{AB} = \tilde{e}^A_{\mu} \tilde{e}^B_{\nu} = -\tilde{e}^B_{\mu} \tilde{e}^A_{\nu} \tilde{e}^\alpha_{\alpha} \tilde{e}^B_{\nu},$$

and using that by construction $\tilde{\omega}^{AB} = -\tilde{\omega}^{BA}$ we obtain $\tilde{e}^B_{\mu} \tilde{e}^A_{\nu} = \tilde{e}^B_{\mu} \tilde{e}^A_{\nu}$ which holds if and only if $\tilde{\Omega}_{\mu\nu\rho} = 0$. Consequently, the manifold $\tilde{M}$ is metric compatible and imposing torsionless condition we can ensure that $\tilde{\Gamma}^\alpha_{\mu\nu}$ in Eq. (4.3) is the Christoffel symbol, namely $\tilde{\Gamma}^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu}(\tilde{g}, \partial \tilde{g})$.

It is worth to point out that, usually, one considers the torsionless condition $\tilde{\omega}^{AB} = -\tilde{\omega}^{BA}$, which is symmetric in $\mu$ and $\nu$.

As an example, let us consider the case where $\tilde{g} = \delta_{\mu\nu}$, which is symmetric in $\mu$ and $\nu$.

$\tilde{\omega}^{AB}$ is metric compatible and imposing torsionless condition we can ensure that $\tilde{\Gamma}^\alpha_{\mu\nu}$ in Eq. (4.3) is the Christoffel symbol, namely $\tilde{\Gamma}^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu}(\tilde{g}, \partial \tilde{g})$. Then, one can check that Eq. (4.8) is a consistency relation which must be satisfied.

As an example, let us consider the case where $\mathcal{M}_D$ and $\tilde{M}_D$ are static spherically symmetric spacetimes, with metrics given by

$$ds^2 = -f^2(r) dt^2 + \frac{1}{h^2(r)} dr^2 + r^2 \gamma_{nm} dx^n dx^m,$$

with $x^a = \phi_1, \ldots, \phi_{D-2}$ and $\gamma_{nm}$ is the metric of a $(D-2)$-dimensional maximally symmetric space. It is direct to show that the manifold $\tilde{M}_D$ has a metric

$$d\tilde{s}^2 = -\tilde{f}^2(r) dt^2 + \frac{1}{h^2(r)} dr^2 + r^2 \gamma_{nm} dx^n dx^m,$$

where

$$\tilde{h} = -\frac{1}{2(D-2)} \left( \frac{1}{r h} \gamma_{nm} \Gamma^n_{nm} - r h \Gamma^n_{nm} \right), \quad \tilde{f} = \int \frac{f h^2}{2h} \left( \frac{f h^2}{f h} \right) dr.$$

Thus, one can calculate the Christoffel symbol $\tilde{\Gamma}^\alpha_{\mu\nu}(\tilde{g}, \partial \tilde{g})$, spin connection $\tilde{\omega}^{AB}$, and vielbein $\tilde{e}^A_{\mu}$ associated with this metric to show that, consistently, the tetrad postulate (4.8) is satisfied. It is also worth to mention that the integration constant that appear after solving the differential equation for $\tilde{f}$ can be easily fixed by demanding that $\mathcal{M} \rightarrow \mathcal{M}_D$ when $\mathcal{M}_D \rightarrow \tilde{M}_D$. In the more specific case where $D = 4$ and $\mathcal{M}_D, \tilde{M}_D$ are respectively the Schwarzschild and Minkowski metrics one gets

$$d\tilde{s}^2 = -dt^2 + \left( 1 - \frac{2M}{r} \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

Having shown that $\tilde{M}$ is not a new independent manifold, because its geometry can be completely characterized with the geometric quantities of $\mathcal{M}$ and $\tilde{M}$, we are now able to give a tensorial version of the CW theorem.

B. Chern-Weil theorem for the hybrid manifold

Given a pair of pseudo-Riemannian manifolds $(\mathcal{M}, \tilde{\mathcal{M}})$ the hybrid connection $\tilde{\omega}$ defined in Eq. (4.0) allows to construct a third auxiliary pseudo-Riemannian manifold $\mathcal{M}$ whose geometry is completely determined in terms of

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6 A similar situation occurs in tensorial language. Given the metric tensor $\tilde{g}_{\mu\nu}$ the metricity condition $\nabla_\lambda \tilde{g}_{\mu\nu} = 0$ represents a set of $D^2 (D + 1) / 2$ algebraic equations to solve the $D^2 (D + 1) / 2$ components of a torsionless connection $\tilde{\Gamma}^\alpha_{\mu\nu}$, which is symmetric in $\mu\nu$. The inverse process would be: Given a set of functions $\tilde{\Gamma}^\alpha_{\mu\nu}$ that transform as an affine connection, the metricity condition can be regarded as a set of partial differential equations to determine $\tilde{g}_{\mu\nu}$. Integrability is ensured by the fact that the symmetric connection $\tilde{\Gamma}^\alpha_{\mu\nu}$ allows to calculate the Riemann tensor $R^\alpha_{\beta\mu\nu} = \partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_\nu \tilde{\Gamma}^\alpha_{\beta\mu} + \tilde{\Gamma}^\alpha_{\beta\rho} \tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\alpha_{\mu\rho} \tilde{\Gamma}^\rho_{\beta\nu}$, which characterizes univocally the geometry of a pseudo-Riemannian manifold $\mathcal{M}$. Therefore, the metric $\tilde{g}_{\mu\nu}$ for which $\tilde{\Gamma}^\alpha_{\mu\nu}(\tilde{g}, \partial \tilde{g})$ is the Christoffel symbol can always be determined (up to a coordinate transformation).
easily written in terms of tensorial objects of where \( M \) for \( \omega \) given by hybrid connection \( \tilde{\omega} \). The l.h.s. of \((4.10)\) can be easily written in terms of tensorial objects of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) while the translation of the r.h.s. using the tetrad postulates for \( \omega \) and \( \tilde{\omega} \) only is problematic (see Eq. \((5.5)\) of the Section \( \text{III A} \)). On the other hand, if we use the definition of the hybrid connection \( \tilde{\omega} \) given by Eq. \((4.10)\), the r.h.s. of \((4.10)\) can be written as a tensorial expression with respect to the pair \( (\mathcal{M}, \tilde{\mathcal{M}}) \) instead of \( (\mathcal{M}, \mathcal{M}) \). Indeed, we have

\[
d(\mathcal{T}(1)(\tilde{\omega})) = d(\varepsilon_{AB}\tilde{\omega}^{AB}) = d(\varepsilon_{AB}\tilde{\omega}^{AB}) = \partial_\mu k^{\mu}_1 d^2 x, \tag{4.11}\]

where \( k^{\mu}_1 = \sqrt{-g}\delta_{\alpha\beta}\Delta_\mu^{\alpha\beta} \) is the Katz vector density in 2 dimensions. Thus, Eq. \((4.10)\) reads

\[
\left(\sqrt{-\tilde{g}}R - \sqrt{-g}\tilde{R}\right) d^2 x = \partial_\mu k^{\mu}_1 d^2 x.
\]

Denoting by \( \mathcal{E}_2[\mathcal{M}] = \sqrt{-\tilde{g}} R d^2 x \) and \( \mathcal{E}_2[\tilde{\mathcal{M}}] = \sqrt{-g} R d^2 x \) the tensorial expression for the topological terms associated with \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), and denoting by \( k^{\mu}_1[\mathcal{M}, \tilde{\mathcal{M}}] \) the vector density \((4.3)\) which depends on the tensorial objects of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), the tensorial version that we have obtained for the Chern-Weil theorem has the following schematical structure,

\[
\mathcal{E}_2[\mathcal{M}] - \mathcal{E}_2[\tilde{\mathcal{M}}(\mathcal{M}, \tilde{\mathcal{M}})] = \partial_\mu (k^{\mu}_1[\mathcal{M}, \tilde{\mathcal{M}}]) d^2 x. \tag{4.12}
\]

This result can be extended for any pair of given 2\(p\)-dimensional Riemannian manifolds \((\mathcal{M}, \tilde{\mathcal{M}})\). After constructing the auxiliary manifold \( \tilde{\mathcal{M}} \) with the hybrid connection \((1.10)\) the Chern-Weil theorem for the pair \((\mathcal{M}, \tilde{\mathcal{M}}) \) is given by

\[
\mathcal{E}_{2p}(\Omega) - \mathcal{E}_{2p}(\tilde{\Omega}) = d[\mathcal{T}^{(2p-1)}(\tilde{\theta}, \Omega, \tilde{\Omega})], \quad \text{with} \quad \tilde{\theta}^{AB} \equiv \omega^{AB} - \tilde{\omega}^{AB}, \tag{4.13}
\]

where

\[
\mathcal{E}_{2p}(\Omega) \equiv \varepsilon_{A_1 \ldots A_{2p}} \Omega^{A_1 A_2} \ldots \Omega^{A_{2p-1} A_{2p}} = \frac{1}{2^p} \sqrt{-g} \delta_{\alpha_1 \ldots \alpha_{2p}} R^{\alpha_1 \alpha_2} \ldots R^{\alpha_{2p-2} \alpha_{2p}} d^2 x, \tag{4.14}
\]

are the corresponding Euler terms and

\[
\mathcal{T}^{(2p-1)}(\tilde{\theta}, \Omega, \tilde{\Omega}) = p \int_0^1 dt \varepsilon_{A_1 \ldots A_{2p}} \tilde{\theta}^{A_1 A_2} \Omega^{A_1 A_2} \ldots \tilde{\Omega}^{A_{2p-1} A_{2p}} \tag{4.15}
\]

is the transgression form with \( \Omega^{AB}(t) = d\omega^{AB} + \omega^A(t) C^{\alpha B}(t) \) and \( \omega^{AB}(t) = \tilde{\omega}^{AB} + t \tilde{\theta}^{AB} \) is a connection which interpolates between \( \tilde{\omega}^{AB} \) and \( \omega^{AB} \).

From the different alternative expressions of the interpolating curvature may have (see \((2.4),(2.10)\)),

\[
\Omega^{AB}(t) = \Omega^{AB} + (t - 1) D \tilde{\theta}^{AB} + (t - 1)^2 \theta C^{\alpha B} \tag{4.16}
\]

is useful to translate the exterior derivative of the transgression \((4.15)\) to tensorial language. Indeed, using Eq. \((4.5)\), the relation \( g_{\gamma \lambda} = \eta_{CD} e^D_{(\gamma} e^C_{\lambda)} \) and

\[
D \tilde{\theta}^{AB} = D_\mu \tilde{\theta}^{AB} dx^\mu dx^\nu = e^A_{\alpha} e^B_{\beta} \nabla_\mu \Delta^{[\alpha\beta]} dx^\mu dx^\nu, \quad \text{with} \quad \Delta^{[\alpha\beta]} = g^{[\gamma \lambda} \Delta_\nu^{\alpha\beta]} \tag{4.17}
\]

which can be proved using the tetrad postulate plus the torsionless condition, we get

\[
\Omega^{AB}_{(t)} = e_{(A}^{A} e^{B)\beta} \left( \frac{1}{2} R_{\mu\nu}^{\beta\mu} + (t - 1) \nabla_\mu \Delta_{[\alpha\beta]} + (t - 1)^2 g_{\gamma \lambda} \Delta_{[\mu}^{[\alpha\beta]} \Delta_{\nu]}^{[\lambda]} \right) dx^\mu dx^\nu. \tag{4.18}
\]
Thus, vielbeins of the same type can be factorized from \( \tilde{\theta}^{-A_1 A_2} \) and each interpolating curvature in (4.15) so the use of the identity (2.8) allows to write

\[
d[\mathcal{I}^{(2p-1)}(\tilde{\theta}, \Omega, \tilde{\Omega})] = \partial_\mu k^\mu_{(p)} d^{2p} x, \tag{4.19}
\]

where

\[
k^\mu_{(p)} = \sqrt{-g} p \int_0^1 dt \tilde{\theta}^{\alpha_1 \alpha_2 \ldots \alpha_{2p}} \Delta_{\alpha_1 \alpha_2} \left( \frac{1}{2} R_{\gamma \delta \alpha_1 \alpha_2} + (t - 1) \nabla_{\gamma \delta} \Delta_{\alpha_1 \alpha_2} + (t - 1)^2 g_{\gamma \delta \lambda \pi} \Delta_{\alpha_1 \alpha_2} \right) \times \ldots
\]

\[
\ldots \times \left( \frac{1}{2} R^{\alpha_1 \alpha_2 \ldots \alpha_{2p-1} \alpha_2} + (t - 1) \nabla_{\alpha_1 \alpha_2 \ldots \alpha_{2p-1} \alpha_2} + (t - 1)^2 g_{\alpha_1 \alpha_2 \ldots \alpha_{2p-1} \alpha_2} \Delta_{\alpha_1 \alpha_2 \ldots \alpha_{2p-1} \alpha_2} \right) \tag{4.20}
\]

is a vector density that is completely characterized by the manifolds \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \), i.e., \( k^\mu_{(p)} = k^\mu_{(p)} [\mathcal{M}, \tilde{\mathcal{M}}] \). Therefore, the tensorial version of the CW theorem that we have constructed can be written as,

\[
\frac{1}{2p} \sqrt{-g} \tilde{\theta}^{\mu_1 \mu_2 \ldots \mu_{2p}} \left( R^\mu_1 \mu_2 \ldots R^{\alpha_1 \alpha_2} - \check{R}^\mu_1 \mu_2 \ldots \check{R}^{\alpha_1 \alpha_2} \right) d^{2p} x = \partial_\mu k^\mu_{(p)} d^{2p} x, \tag{4.21}
\]

which has the following structure

\[
\mathcal{E}_{2p} [\mathcal{M}] - \mathcal{E}_{2p} [\mathcal{M}, \tilde{\mathcal{M}}(\mathcal{M}, \mathcal{M})] = \partial_\mu (k^\mu_{(p)} [\mathcal{M}, \tilde{\mathcal{M}}]) d^{2p} x, \tag{4.22}
\]

where \( \mathcal{E}_{2p} [\mathcal{M}] \) and \( \mathcal{E}_{2p} [\tilde{\mathcal{M}}] \) denotes the tensorial expression of the Euler terms (4.14).

Eq. (4.20, 4.22) is the result we aimed at: to write the Chern-Weil theorem is terms of purely spacetime tensorial quantities. As we have seen in detail, this requires the explicit introduction of a background manifold, \( \mathcal{M} \) whose role is (1) to construct the spacetime tensors \( \Delta = \Gamma - \Gamma \) which are essential in the definition of the vector \( k^\mu_{(p)} \); see (4.20); (2) to construct the manifold \( \tilde{\mathcal{M}} \) such that the divergence of the vector \( k^\mu_{(p)} \) is the difference of the topological terms of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \).

In the case we choose \( \tilde{\mathcal{M}} \) to be a product manifold (whose metric can be written as \( ds^2 = dx^2 + \tilde{h}(x) dx^2 dx^2 \) so that the extrinsic curvatures of the \( r = \text{const} \) hypersurfaces are zero) its topological term vanishes (see [13]), as well as that of \( \mathcal{M} \) (for the same reasons, as can be easily shown from of the definition of the hybrid connection (4.9)). This shows explicitly that the Einstein-Hilbert, Gauss-Bonnet and in general the Lovelock terms reduce, in the critical dimension \( D = 2p \), to the divergence of a vector density constructed with spacetime tensors, \( \mathcal{E}_{2p} [\mathcal{M}] = \partial_\mu k^\mu_{(p)} \). If the product manifold \( \tilde{\mathcal{M}} \) is also cobordant, which means that a specific surface \( r = r_0 \) coincides with the boundary of \( \mathcal{M} \), then the component \( k^\mu_{(p)} \) normal to the boundary coincides with Myers’ boundary term (see [13]). However, this does not mean that there is no underlying structure, since the vector \( k^\mu_{(p)} \) still depends explicitly on \( \mathcal{M} \). In the generic case, the topological term of \( \mathcal{M} \) does not vanish and represents a topological obstruction to write the Euler term of \( \mathcal{M} \) as a divergence of the vector density \( k^\mu_{(p)} \).

In all cases (whether \( \mathcal{M} \) is an arbitrary background or product manifold) the tensorial translation of the Chern-Weil theorem requires the introduction of an extra structure \( \mathcal{M} \). In fact this should not come as a surprise: for example, Horndeski, in his proof [8] that \( \mathcal{E}_{2p} [\mathcal{M}] \propto \partial_\mu V^\mu_{(p)} \), where

\[
V^\mu_{(p)} = V^\mu_{(p)} (g_{\mu \nu}, R^\mu_{\nu a b}, U^\mu, \nabla_\mu U^\mu), \tag{4.23}
\]

depends also on an extra structure, namely \( U^\mu \) which is an arbitrary non-null contravariant vector (see the explicit expression of \( V^\mu_{(p)} \) in Appendix A).

V. FINAL REMARKS

Given two pseudo-Riemannian manifolds \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) we have introduced an auxiliary manifold \( \tilde{\mathcal{M}} \) whose geometry is completely determined by the first two and that allows to construct the tensorial version of the Chern-Weil theorem (4.21). This expression states that the difference of the Euler terms of \( \mathcal{M} \) and \( \mathcal{M}(\mathcal{M}, \tilde{\mathcal{M}}) \) is the divergence of the vector density (4.20) which is constructed with objects that are tensorial with respect to \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \).

As we will see in [32] (see also [12] for the Gauss-Bonnet case), the tensorial version of the CW theorem presented in this work is the one that must be used (together with a dimensional continuation procedure) to generalize the
procedure developed by Katz, Bicak and Lynden-Bell (KBL) \[12,31\] to calculate conserved charges in a generic Lovelock theory. There the manifolds \( \mathcal{M} \) and \( \mathcal{M} \) are interpreted as the dynamical and background manifolds, while the hybrid manifold \( \mathcal{M} \) is just an auxiliary manifold allowing us to write the KBL vector in the vielbein formalism and to give a proof for the Dirichlet problem in Lovelock gravity.

VI. ACKNOWLEDGMENT

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Appendix A: Myers’ terms from Horndeski divergences

Myers’ boundary terms \[9\] in the critical dimension \( D = 2p \) are given by

\[
I_{\text{Myers}} = \int_{\partial \mathcal{M}_D} d^{2p-1}x \beta^{(p)},
\]

with

\[
\beta^{(p)} = -2\sqrt{-h} \int_0^1 dt \delta^{j_1 \ldots j_{2p-1}} K_{j_1}^{i_1} \left( \frac{1}{2} R^{i_2 j_2}_{j_1 j_1} - t^2 K_{j_2}^{j_1} K_{j_1}^{i_2} \right) \times \cdots \times \left( \frac{1}{2} R^{i_{2p-2} j_{2p-2}}_{j_{2p-2} j_{2p-2}} - t^2 K_{j_{2p-2}}^{j_{2p-2}} K_{j_{2p-2}}^{i_{2p-2}} \right),
\]

where \( R^{ijkl} = R^{ij}_{kl} (h) \) is the intrinsic curvature of the boundary, \( R^{ijkl} = R^{ij}_{kl} (g) \) are the boundary components of the bulk curvature and the coefficients in the last equality comes after performing the integration in the parameter \( t \) (further details can be found, e.g., in Ref. \[13\]). In particular, the double factorial is defined as

\[
n!! = \begin{cases} 
\prod_{k=1}^{n/2} (2k) & , \text{for } n \text{ even}, \\
\prod_{k=1}^{(n+1)/2} (2k-1) & , \text{for } n \text{ odd}.
\end{cases}
\]

On the other hand, in Ref. \[8\] Horndeski has shown explicitly that in the critical dimensions the Lovelock densities are given by a divergence, namely

\[
2^p \int_{\mathcal{M}_D} d^{2p}x E_{2p} [\mathcal{M}] = \int_{\mathcal{M}_D} d^{2p}x \sqrt{-g} \delta^{\mu_1 \ldots \mu_{2p}} R^{\mu_1 \nu_1}_{\mu_2 \nu_2} \ldots R^{\mu_{2p-1} \nu_{2p-1}}_{\mu_{2p} \nu_{2p}} = \int_{\mathcal{M}_D} d^{2p}x \partial_\mu V_{(p)}^\mu,
\]

with \( V_{(p)}^\mu \) being the following vector density

\[
V_{(p)}^\mu = -g \sum_{k=0}^{p-1} C_k \delta^{\mu \nu_1 \ldots \nu_{2p-1}} \partial_\mu U^{\nu_1} \ldots \partial_{\nu_{2k+1}} U^{\nu_{2k+1}} \ldots \partial_{\nu_{2k+3}} U^{\nu_{2k+3}} \ldots R^{\nu_{2p-2} \nu_{2p-1}}_{\nu_{2p-1} \nu_{2p-1}},
\]

where \( \rho = U_\mu U^\mu \), \( C_0 = -4p \) and

\[
C_k = -4^{k+1} \rho \prod_{q=0}^{k-1} \left( \frac{p-q-1}{2q+3} \right).
\]

Here we show that if we use a radial foliation, with Gauss normal coordinates given by \( ds^2 = dr^2 + h_{ij} (r, x^i) \) \( dx^i dx^j \) and if we chose the arbitrary vector \( U^\mu \) to be the normal vector of the surfaces \( r = \text{const} \), namely \( U^\mu = \left( 1, \mathbf{0} \right) \), then
the Eq. [A1] reproduces the Myers boundary term as
\[ \int_{\mathcal{M}_D} d^{2p} x \partial_\mu V^\mu_{(p)} = 2^p I_M \text{e}_r. \]
For doing so, we first write the Myers term as
\[ 2^p I_M = -p \int_{\partial \mathcal{M}_D} d^{2p-1} x \sqrt{-h} \sum_{s=0}^{p-1} \frac{4^{p-s} (p-1)!}{s! (2p - 2s - 1)!} \delta^{j_1 \ldots j_{2p-2}}_{1 \ldots (2p-2)} R^{i_1 i_2} \ldots R^{j_{2s-1} j_{2s}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots} \]
\[ = \int_{\partial \mathcal{M}_D} d^{2p-1} x \sqrt{-h} [-4p \delta^{j_1 \ldots j_{2p-1}}_{1 \ldots (2p-1)} R^{i_1 i_2} \ldots R^{j_{2s-2} j_{2s-1}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots}] \]
\[ = \sum_{s=0}^{p-2} \frac{4^{p-s} (p-1)!}{s! (2p - 2s - 1)!} \delta^{j_1 \ldots j_{2p-2}}_{1 \ldots (2p-2)} R^{i_1 i_2} \ldots R^{j_{2s-1} j_{2s}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots} \] (A2)

Now, using the Gauss theorem in this radial foliation, we can write the Horndeski term as
\[ \int_{\mathcal{M}_D} d^{2p} x \partial_\mu V^\mu_{(p)} = \int_{\partial \mathcal{M}_D} d^{2p-1} x U^I v^\mu = \int_{\partial \mathcal{M}_D} d^{2p-1} x V^\mu_{(p)} \]
\[ = \int_{\partial \mathcal{M}_D} d^{2p-1} x \sqrt{-h} \sum_{k=0}^{p-1} C_k \delta^{i_1 \ldots i_{2p-1}}_{1 \ldots (2p-1)} R^{i_1 i_2} \ldots R^{i_{2s-2} i_{2s-1}} K^{i_{2s+1} + \ldots} \ldots K^{i_{2p-1} + \ldots} \]
where \( V^\mu_{(p)} \) has been calculated using \( \rho = 1, U^I = 1, \delta^{j_1 \ldots j_{2p-1}}_{1 \ldots (2p-1)} = \delta^{j_1 \ldots j_{2p-1}}_{1 \ldots (2p-1)} \) and \( \nabla_j U^I = K^I_j \). Rearranging the indices, the Horndeski term can we rewritten as
\[ \int_{\mathcal{M}_D} d^{2p} x \partial_\mu V^\mu_{(p)} = \int_{\partial \mathcal{M}_D} d^{2p-1} x \sqrt{-h} \sum_{a=0}^{p-s-1} C_{p-s-1} \delta^{j_1 \ldots j_{2p-2}}_{1 \ldots (2p-2)} R^{i_1 i_2} \ldots R^{j_{2s-1} j_{2s}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots} \]
\[ = \int_{\partial \mathcal{M}_D} d^{2p-1} x \sqrt{-h} [-4p \delta^{j_1 \ldots j_{2p-2}}_{1 \ldots (2p-2)} R^{i_1 i_2} \ldots R^{j_{2s-3} j_{2s-2}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots}] \]
\[ = \sum_{a=0}^{p-s-2} \frac{4^{p-s} (p-1)!}{q! (2p - 2q - 1)!} C_{p-s-1} \delta^{j_1 \ldots j_{2p-2}}_{1 \ldots (2p-2)} R^{i_1 i_2} \ldots R^{j_{2s-3} j_{2s-2}} K^{j_{2s+1} + \ldots} \ldots K^{j_{2p-1} + \ldots} \] (A3)
where one can directly check that the first terms in [A2] and [A3] coincide, while the rest of the terms coincides because
\[ C_{p-s-1} = -4^{p-s} p \prod_{q=0}^{p-s-2} \frac{(p - q - 1)}{2q + 3} = -4^{p-s} p \frac{(p - 1)!}{s! (2p - 2s - 1)!}. \]

**Q.E.D.**

**Appendix B: Transgression forms**

Let \( G = \{ T_M \} \) be a Lie algebra and \( A = A^\mu_M T_M dx^\mu \) a Lie valued one-form gauge connection. This means that under a gauge transformation characterized by a group element \( g = \text{exp} (g^M T_M) \) (the parameters \( g^M \) being coordinates in the Lie group manifold \( G \)) the connection transforms as
\[ A \rightarrow g A g^{-1} + g d g^{-1}. \]
If \( Q \) is a \( r \)-form transforming as \( Q \rightarrow g Q g^{-1} \) under a gauge transformation, then the covariant derivative with respect to \( A \) is defined by \( DQ = dQ + [A, Q] \) (with \( [\cdot, \cdot] \) being the commutator) and transforms as \( DQ \rightarrow g (DQ) g^{-1} \).

The strength field two-form is defined as \( F = dA + AA \), where for shortness we omit the wedge product between differential forms. Thus, by construction the strength field transform as \( F \rightarrow g F g^{-1} \) and satisfies the Bianchi identity, \( D F = 0 \). It is also direct to show that the symmetrized trace of a given strength field power, denoted by \( (F^p) \), is invariant under gauge transformations. This can be directly shown using the properties of the symmetrized the trace and the wedge product. Hence, \( (F^p) \) is usually called a **topological term**.
A transgression form is defined by the Chern-Weyl theorem, which states that if \( A \) and \( \tilde{A} \) are two gauge connections valued on the same algebra with strength fields \( F \) and \( \tilde{F} \), then \( \langle F^p \rangle \) and \( \langle \tilde{F}^p \rangle \) are closed forms, i.e.,

\[
d \langle F^p \rangle = d \langle \tilde{F}^p \rangle = 0
\]

and

\[
\langle F^p \rangle - \langle \tilde{F}^p \rangle = d T^{(2p-1)}(A, \tilde{A})
\]

where

\[
T^{(2p-1)}(A, \tilde{A}) = p \int_0^1 dt \langle \theta F^p_{t-1} \rangle,
\]

with \( \theta = A - \tilde{A} \), \( F_1 = dA + A_A \), and where \( A_A = \tilde{A} + t\theta \) is a connection interpolating between \( \tilde{A} \) and \( A \). Eq. (B1) states that both topological terms are closed forms, whilst Eq. (B2) tells that their difference is an exact form defined in (B3) by the \((2p-1)\)-form \( T^{(2p-1)} \), which is known as a transgression form.

In this construction, we frequently use the property \( d \langle Q \rangle = \langle DQ \rangle \) when \( Q \) is a covariant object transforming as \( Q \to gQg^{-1} \). Using \( D\theta = D\tilde{\theta} - 2\theta^2 \) and \( F_1 = F \), the interpolating curvature can be written in the following three alternative forms,

\[
F_1 = F + t\tilde{\theta} + t^2 \theta^2,
\]

\[
= F + (t - 1) D\tilde{\theta} + (t^2 - 2t + 1) \tilde{\theta}^2,
\]

\[
= tF + (1 - t) F - t (1 - t) \theta^2.
\]

Let us consider now the case where the symmetry is given by the Lorentz algebra with generators \( \{ T_M \} = \{ J_{AB} \} \) satisfying,

\[
[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}.
\]

In this case we have,

\[
A = \frac{1}{2} \omega^{AB} J_{AB}, \quad \tilde{A} = \frac{1}{2} \tilde{\omega}^{AB} J_{AB}, \quad \theta = \frac{1}{2} \tilde{\theta}^{AB} J_{AB},
\]

\[
F = \frac{1}{2} Q^{AB} J_{AB}, \quad \tilde{F} = \frac{1}{2} \bar{Q}^{AB} J_{AB},
\]

where \( \omega^{AB} \) and \( \tilde{\omega}^{AB} \) are two Lorentz spin connections and \( \theta^{AB} = \omega^{AB} - \tilde{\omega}^{AB} \). The topological terms can be written as,

\[
\langle F^p \rangle = \frac{1}{2^p} \varepsilon_{A_1 \cdots A_{2p}} \Omega^{A_1 A_2} \cdots \Omega^{A_{2p-1} A_{2p}} = \frac{1}{2^p} \varepsilon_{2p}(\Omega),
\]

\[
\langle \tilde{F}^p \rangle = \frac{1}{2^p} \varepsilon_{A_1 \cdots A_{2p}} \tilde{\Omega}^{A_1 A_2} \cdots \tilde{\Omega}^{A_{2p-1} A_{2p}} = \frac{1}{2^p} \varepsilon_{2p}(\bar{\Omega}),
\]

where \( \varepsilon_{A_1 \cdots A_{2p}} = \{ J_{A_1 A_2} \cdots J_{A_{2p-1} A_{2p}} \} \) and where \( \varepsilon_{2p}(\Omega), \varepsilon_{2p}(\bar{\Omega}) \) are called Euler topological terms. Using \( \tilde{\theta}^2 = \frac{1}{2} \tilde{\theta}^{AC} [g C] J_{AB} \), which can be shown using (B7), the interpolating strength field \( F_1 = \frac{1}{4} Q^{AB} J_{AB} \) can be written in the following alternative forms,

\[
\Omega^{AB}_{(t)} = \Omega^{AB} + t D\tilde{\theta}^{AB} + t^2 \tilde{\theta}^{AC} [g C] \tilde{\theta}^{CB},
\]

\[
= \Omega^{AB} + (t - 1) D\tilde{\theta}^{AB} + (t - 1)^2 \tilde{\theta}^{AC} [g C] \tilde{\theta}^{CB}.
\]

\[
= t \Omega^{AB} + (1 - t) \tilde{\theta}^{AB} + (1 - t)^2 \tilde{\theta}^{AC} [g C] \tilde{\theta}^{CB}.
\]

Thus, the transgression form is given by \( T^{(2p-1)}(\omega, \bar{\omega}) = \frac{1}{2^p} T^{(2p-1)}(\omega, \bar{\omega}) \), where

\[
T^{(2p-1)}(\omega, \bar{\omega}) = p \int_0^1 dt \varepsilon_{A_1 \cdots A_{2p}} \tilde{\theta}^{A_1 A_2} \Omega^{A_3 A_4}_{(t)} \cdots \Omega^{A_{2p-1} A_{2p}}_{(t)},
\]

so the Chern-Weyl theorem for the Lorentz symmetry reads,

\[
\varepsilon_{2p}(\Omega) - \varepsilon_{2p}(\bar{\Omega}) = d T^{(2p-1)}(\omega, \bar{\omega}).
\]
Appendix C: Tensorial version of the CW theorem in $D=2$

In order to formulate the two-dimensional Chern-Weil theorem in tensorial language, one can use the fact that all 2-dimensional pseudo-Riemannian manifolds are conformally related. To do this, let us consider a given metric $g^{(0)}_{\mu\nu}$ and two functions $u(x^\mu)$ and $\bar{u}(x^\mu)$ such that the metrics of the manifolds $\mathcal{M}_2$ and $\bar{\mathcal{M}}_2$ are given by

$$g_{\mu\nu} = e^{2u}g^{(0)}_{\mu\nu}, \quad \bar{g}_{\mu\nu} = e^{2\bar{u}}g^{(0)}_{\mu\nu}. \quad \text{(C1)}$$

A direct calculation of the Ricci tensors leads the following relation (in $D=2$)

$$R^{(0)}_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu} \Box u = \bar{R}_{\mu\nu} + \bar{g}_{\mu\nu} \Box \bar{u}. \quad \text{(C2)}$$

Using $g^{(0)}_{\mu\nu} = e^{2u}g_{\mu\nu} = e^{2\bar{u}}\bar{g}_{\mu\nu}$ we obtain the following relation between the Ricci scalars,

$$R^{(0)} = e^{2u}(R + 2\Box u) = e^{2\bar{u}}(\bar{R} + 2\Box \bar{u}). \quad \text{(C3)}$$

Thus, with the usual properties of the operators $\Box$ and $\bar{\Box}$ the last relation can be equivalently written as,

$$R = -2\Box u + e^{-2u}R^{(0)} = -\frac{2}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}g^{\mu\nu} \partial_\nu u \right) + e^{-2u}R^{(0)}; \quad \text{(C4)}$$

Then, the difference of the topological terms $\sqrt{-g}Rd^2x$ and $\sqrt{-\bar{g}}\bar{R}d^2x$ is given by

$$(\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R})d^2x = \left[ -2\partial_\mu \left( \sqrt{-g}g^{\mu\nu} \partial_\nu u - \sqrt{-\bar{g}}\bar{g}^{\mu\nu} \partial_\nu \bar{u} \right) + \sqrt{-g}e^{-2u}R^{(0)} - \sqrt{-\bar{g}}e^{-2\bar{u}}R^{(0)} \right] d^2x. \quad \text{(C5)}$$

Finally, using the relations

$$\sqrt{-g} = e^{2u}\sqrt{-g^{(0)}}, \quad \sqrt{-\bar{g}} = e^{2\bar{u}}\sqrt{-g^{(0)}}, \quad g_{\mu\nu} = e^{2U}g_{\mu\nu}, \quad \bar{g}_{\mu\nu} = e^{2\bar{U}}\bar{g}_{\mu\nu}, \quad U = u - \bar{u}, \quad \text{(C6)}$$

that can be easily derived from (C1), we get

$$(\sqrt{-g}R - \sqrt{-\bar{g}}\bar{R})d^2x = -\partial_\mu v^\mu d^2x, \quad \text{(C7)}$$

where

$$v^\mu = 2\sqrt{-g}g^{\mu\nu} \partial_\nu U. \quad \text{(C8)}$$

Eq. (C7), with $v^\mu$ given by (C8), represents a tensorial version of the Chern-Weil theorem which is free of objects coming from the vielbein formalism and that depends only on the metrics $g_{\mu\nu}, \bar{g}_{\mu\nu}$ and the conformal factor $U$ relating the metrics of $\mathcal{M}_2$ and $\bar{\mathcal{M}}_2$. Although this is an interesting result, which is valid for any given pair of manifold $\mathcal{M}_2$ and $\bar{\mathcal{M}}_2$, a generalization to higher dimensions is not possible because the fact that all metrics are conformally equivalent is an accident that happens only in $D=2$.

A generalization to $D=2p$ might work only under the assumption that $\mathcal{M}_2$ and $\bar{\mathcal{M}}_2$ are conformally equivalent. We leave that problem for a possible future work.

Appendix D: Relation between vielbeins of different spaces

The orthotormal inverse vielbeins $e^\mu_A$ at $P \in \mathcal{M}_D$ and $\bar{e}^\mu_A$ at $\bar{P} \in \bar{\mathcal{M}}_D$, can be defined by means of two different coordinate transformations, one in $P$ the other at $\bar{P}$,

$$x^\mu = x^\mu (y^A) \quad \text{and} \quad \bar{x}^\mu = \bar{x}^\mu (\bar{y}^A), \quad \text{(D1)}$$

so that the metrics in $\mathcal{M}_D$ and $\bar{\mathcal{M}}_D$ become Minkowski at $P$ and $\bar{P}$,

$$e^\mu_A e^\nu_B g_{\mu\nu} = \eta_{AB}, \quad \text{with} \quad e^\mu_A (P) = \frac{\partial x^\mu}{\partial y^A} (P), \quad \text{(D2)}$$

$$\bar{e}^\mu_A \bar{e}^\nu_B \bar{g}_{\mu\nu} = \eta_{AB}, \quad \text{with} \quad \bar{e}^\mu_A (\bar{P}) = \frac{\partial \bar{x}^\mu}{\partial \bar{y}^A} (\bar{P}). \quad \text{(D3)}$$
The coordinates \( y^A \) are \( \tilde{y}^A \) must be different, otherwise we are led to the contradiction \( g_{\mu\nu} = \tilde{g}_{\mu\nu} \). However, the existence of the mapping \( \sigma \) introduced in Section [III] implies that \( y^A \) and \( \tilde{y}^A \) are smoothly related. Indeed, using the relations (D1) are invertible, we can write

\[
y^A = y^A \left( x^\mu \right) = y^A \left( x^\mu \left( \tilde{y}^B \right) \right) \equiv y^A \left( x^\mu, \tilde{y}^B \right) .
\] (D4)

In the last expression we wrote a explicitly a dependence on the coordinate \( x^\mu \). The reason is that the relations (D1) are not simple coordinate transformations made in a given patch of each manifold. They are rather one pair of coordinate transformation for each couple of points \( (P, \tilde{P}) \). As \( y^A \) and \( \tilde{y}^A \) are cartesian coordinates of two Minkowski spaces tangent to the points \( P \) and \( \tilde{P} \) (i.e., such that the metric in both cases is \( \eta_{AB} \)) we see that \( y^A \) must be a linear, point dependent function of \( \tilde{y}^A \). Without loss of generality we can assume that is also homogeneous, namely \( y^A = K^A_B \left( x \right) \tilde{y}^B \), so the vielbeins are related by

\[
e^A = K^A_B \left( x \right) e^B .
\] (D5)

Thus, for any given pair of manifolds \( \mathcal{M}_D \) and \( \tilde{\mathcal{M}}_D \), the matrix \( K \) can be directly solved from (D5) as

\[
K^A_B \left( x \right) = e^A_\mu \left( x \right) \tilde{e}^B_\nu \left( x \right) .
\]

Clearly, the matrix \( K^A_B \left( x \right) \) cannot be a Lorentz rotation, otherwise \( ds^2 \) and \( ds^2_\tilde{ } \) would coincide. To understand better this result, we remark that in the coordinates \( y^A \), the metric of the Minkowski tangent space \( T_P \left( \mathcal{M}_D \right) \) in \( P \) is given by \( \eta_{AB} \). If we use the coordinates \( \tilde{y}^A = \left( K^{-1} \right)^A_B \left( x \right) y^B \), then the metric of \( T_P \left( \tilde{\mathcal{M}}_D \right) \) is given by

\[
g_{AB} \left( x \right) = K^C_A \left( x \right) K^D_B \left( x \right) \eta_{CD} \neq \eta_{AB} ,
\] (D6)

i.e., the Lorentz metric \( \eta_{AB} \) is not preserved because the considered coordinate transformation is not of the Lorentz type. The same applies for the tangent space \( T_P \left( \tilde{\mathcal{M}}_D \right) \), whose metric in the coordinates \( \tilde{y}^A \) is given by \( \eta_{AB} \) while in coordinates \( y^A = K^A_B \left( x \right) \tilde{y}^B \) it is given by,

\[
\tilde{g}_{AB} \left( x \right) = \left( K^{-1} \right)^C_A \left( x \right) \left( K^{-1} \right)^D_B \left( x \right) \eta_{CD} \neq \eta_{AB} .
\] (D7)

As an example, for the case of static spherically symmetric manifolds with metrics,

\[
ds^2 = -f^2 \left( r \right) dt^2 + \frac{1}{h^2 \left( r \right)} dr^2 + r^2 d\Omega^2_{D-2} , \quad ds^2_\tilde{ } = -f^2 \left( \tilde{r} \right) d\tilde{t}^2 + \frac{1}{\tilde{h}^2 \left( \tilde{r} \right)} d\tilde{r}^2 + \tilde{r}^2 d\tilde{\Omega}^2_{D-2} ,
\] (D8)

we obtain,

\[
K = \left( K^A_B \right) = \begin{pmatrix} f / \tilde{f} & 0 & 0 \\ 0 & h / \tilde{h} & 0 \\ 0 & 0 & \delta^a_b \end{pmatrix} ,
\]

with \( a, b = 2, \ldots, D - 1 \). Thus, we see explicitly that \( K^T \eta K \neq \eta \) unless \( f = \tilde{f} \) and \( h = \tilde{h} \).

**Appendix E: Geometric properties of \( \tilde{\omega}^A_B \)**

Let us associate \( \tilde{\omega}^A_B \) with an auxiliary manifold \( \tilde{\mathcal{M}} \) with metric \( \tilde{g}_{\mu\nu} \) and affine connection \( \tilde{\Gamma}^\alpha_{\mu\gamma} \) so that the vielbein \( \tilde{e}^A \) and the hybrid spin connection \( \tilde{\omega}^A_B \) verify the usual relations

\[
\tilde{g}_{\mu\nu} = \tilde{e}^A_\mu \tilde{e}^B_\nu \eta_{AB} , \quad \tilde{\omega}^A_B = \tilde{e}^A_\alpha \tilde{\Gamma}^\alpha_{\mu\gamma} + \tilde{e}^A_\alpha \partial_\mu \tilde{e}^\alpha_B ,
\] (E1)

and allow to define the curvature and torsion two-forms as

\[
\tilde{\Omega}^A_B \equiv \partial \tilde{\omega}^A_B + \tilde{\omega}^A_C \tilde{\omega}^C_B , \quad \tilde{T}^A \equiv \partial \tilde{e}^A .
\] (E3)

\footnote{We also choose a smooth mapping allowing us to use the same coordinates \( x^\mu \) for each point \( P \in \mathcal{M}_D \) and \( \tilde{P} \in \tilde{\mathcal{M}}_D \).}
A direct consequence of the definition (4.1) is that $\tilde{M}$ is not a metric compatible manifold, i.e., $\tilde{\nabla}g_{\mu\nu} \neq 0$ and thus $\tilde{\Gamma}_{\mu\nu}^\alpha$ is not the Christoffel symbol. To see this, we first notice that $\tilde{\omega}^{AB}$ is not antisymmetric (or, equivalently $\tilde{\nabla}^2\eta_{AB} \neq 0$). Indeed, Eq. (4.1) can be written as $\tilde{\omega}_{\mu}^{AB} = e_A^\alpha \nabla_{\mu} e^{B\alpha} = -e^{B\alpha} \nabla_{\mu} e_A^{\alpha}$ and using $\nabla_{\lambda} g_{\mu\nu} \neq 0$ (with $\nabla_{\lambda}$ being the covariant derivative with respect to $\Gamma_{\mu\nu}^\alpha$) one gets
\begin{equation}
\tilde{\omega}_{\mu}^{AB} = -e^{B\alpha} \nabla_{\mu} e_A^{\alpha} \neq -e_{\alpha}^{B} \nabla_{\mu} e^{A\alpha} = -\tilde{\omega}_{\mu}^{BA} .
\end{equation}
Similarly, Eq. (4.2) can be rewritten as $\tilde{c}_{\mu}^{AB} = e_A^\alpha \nabla_{\mu} e^{B\alpha} = -e^{B\alpha} \nabla_{\mu} e_A^{\alpha}$ and then, using $\tilde{\omega}_{\mu}^{AB} \neq -\tilde{\omega}_{\mu}^{BA}$ we have
\begin{equation}
\tilde{c}_{\mu}^{B\alpha} \nabla_{\mu} e_A^{\alpha} \neq \tilde{c}_{\mu}^{B\alpha} \nabla_{\mu} e^{A\alpha} ,
\end{equation}
which clearly implies $\tilde{\nabla}g_{\mu\nu} \neq 0$.

The non metricity of $\tilde{M}$ does not necessarily represent a problem, because it can be thought just as an auxiliary manifold allowing to translate Lorentz invariant expressions constructed with $\tilde{\theta}^{AB}$, to tensorial language. Then, is necessary to show that $\tilde{M}$ carries no new independent information, i.e., that its geometry can be completely fixed in terms of geometrical quantities of $M$ and $\tilde{M}$. Indeed, from the definition (4.1) the components $\tilde{\omega}^{AB}_{\mu}$ can be completely solved in terms of geometrical quantities of $M$ and $\tilde{M}$ and plugging this in (4.2) together with the torsionless condition, $\tilde{\Gamma}_{\mu\nu}^\alpha = \tilde{\Gamma}_{\nu\mu}^\alpha$, one can solve the independent components of $\tilde{c}_{\mu}^{AB}$ and $\tilde{\Gamma}_{\nu\mu}^\alpha$. Even if this is a hard task, due to the big number of unknown functions that must be solved when $\tilde{\Gamma}_{\nu\mu}^\alpha$ is not the Christoffel symbol, it can always be done.

The real problem about the hybrid connection (4.1) is related with the definition of the curvature made in (E3). Even if the Bianchi identities $\tilde{\nabla}^\alpha \tilde{\eta}_{\mu\nu} = 0$ and $\tilde{\nabla}^\alpha \tilde{\eta}_{\mu\rho\nu} = 0$ are satisfied, for
\begin{equation}
\tilde{\eta}_{\mu\nu}^{AB} \equiv \tilde{\eta}^{BC} \tilde{\eta}_{\mu}^{AC} = d\tilde{\omega}^{AB} + \tilde{\omega}^{AC} \tilde{\omega}_{\nu}^{CB}
\end{equation}
we have instead
\begin{equation}
\tilde{\nabla}^\alpha \tilde{\eta}_{\mu\nu}^{AB} = d\tilde{\omega}^{A[B} + \tilde{\omega}^{A[C} \tilde{\omega}_{\nu}^{CB],}
\end{equation}
which leads
\begin{equation}
\tilde{\eta}_{\mu\nu}^{AB} = d\tilde{\omega}^{[AB]} + \eta_{CD} \tilde{\omega}^{[AC]} \tilde{\omega}^{[DB]},
\end{equation}
which is clearly different from (E5). Defined by (E8), $\tilde{\eta}_{\mu\nu}^{AB}$ is antisymmetric by construction and it should be regarded as the strenght field for $\tilde{\omega}_{\mu}^{AB}$ rather than $\tilde{\omega}^{AB}$. Indeed, it is direct to show that under a Lorentz transformation
\begin{equation}
\tilde{\omega}_{\mu}^{[AB]} = \Lambda^{A'}_{C} \Lambda^{B'}_{D} \tilde{\omega}_{\mu}^{[CD]} + \eta_{CD} \Lambda^{A'}_{C} \tilde{\omega}^{B'}_{\mu} ,
\end{equation}
and thus, $\tilde{\omega}_{\mu}^{[AB]}$ is also a well-defined Lorentz connection. In addition, the Bianchi identity coming from the gauge formulation is then given by $\tilde{\nabla}^\alpha \tilde{\eta}_{\mu\nu}^{AB} = 0$ (where $\tilde{\nabla}$ denotes the covariant derivative with respect to the gauge field $\tilde{A}$), and leads
\begin{equation}
\tilde{\nabla}^\alpha \tilde{\eta}_{\mu\nu}^{AB} = d\tilde{\omega}^{A[B} + \eta_{CD} \tilde{\omega}^{[AC]} \tilde{\omega}^{[DB]} + \eta_{CD} \tilde{\omega}^{[BC]} \tilde{\omega}_{\nu}^{[AD]} = 0 .
\end{equation}
To avoid antisymmetrization brackets, we can define the antisymmetric hybrid spin connection as:

$$\tilde{\omega}_{\mu}^{AB} \equiv \varepsilon^{[AB]} \tilde{\omega}_{\mu} = \frac{1}{2} \left( e_\alpha ^A e_{\beta} ^B \Gamma_\mu ^\alpha _\beta + e_\beta ^A e_{\alpha} ^B \Gamma_\mu ^\beta _\alpha - e_\alpha ^B e_{\beta} ^A \Gamma_\mu ^\alpha _\beta - e_\beta ^B e_{\alpha} ^A \Gamma_\mu ^\beta _\alpha \right).$$ (E10)

The strenght field $\tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = \frac{1}{8} \omega^{AB} J_{AB}$ associated with the gauge connection $\tilde{A} = \frac{1}{8} \omega^{AB} J_{AB}$ lead the following definition for Lorentz curvature two-form $\tilde{\Omega}^{AB} \equiv d\tilde{\omega}^{AB} + \tilde{\omega}^{CA} \tilde{\omega}_B ^C$, with $\tilde{\omega}^A_B = \eta_{BC} \tilde{\omega}^{AC}$, while the Bianchi identity $D \tilde{F} \equiv d\tilde{F} + [\tilde{A} , \tilde{F}] = 0$ reads $D \tilde{\Omega}^A_B = d\tilde{\Omega}^A_B + \tilde{\omega}^{CA} \tilde{\Omega}_B ^C + \tilde{\omega}^C_B \tilde{\Omega}^{AC} = 0$. Using $\tilde{A}^A_B \equiv \eta_{BC} \tilde{\Omega}^{AC}$ leads the usual expression for the Lorentz curvature

$$\tilde{\Omega}^A_B = d\tilde{\omega}^A_B + \tilde{\omega}^C_B \tilde{\omega}_B ^C,$$ (E11)

and then the Bianchi identity can also be written as $D \tilde{\Omega}^A_B = 0$, because $D \tilde{\Omega}^A_B = 0$ due to the fact that $\tilde{\omega}$ is antisymmetric by construction.

Due to the antisymmetry of the Lorentz generators $J_{AB}$ one sees that $\tilde{A} = \tilde{A}$ and $\tilde{F} = \tilde{F}$. Thus, the introduction of (E10) can be thought just as a change of notation that allows to write the expressions (E11) which is free of antisymmetrization brackets.

However, in Section [XX] this notation has proved to be very useful to determine the geometrical properties of the auxiliary pseudo-Riemannian manifold $\mathcal{M}$ that allowed us to give a tensorial formulation of the CW theorem.

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