A Note on T-dualities in the Pure Spinor Heterotic String

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In this note we study the preservation of the classical pure spinor BRST constraints under super T-duality transformations. We also determine the invariance of the one-loop conformal invariance and of the local gauge and Lorentz anomalies under the super T-dualities.

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1. Introduction

Dualities in string theory have become a very important property in string theory. They allow to show equivalence between different types of strings implying, in this way, a unifying criteria in string theory. One of these dualities is the target (T-)duality which makes type IIA equivalent to type IIB [1]. From the world-sheet point of view, T-dualities imply equivalence between different types of string theory backgrounds following the Buscher procedure [2]. The idea is to consider a background independent of some direction and then introduce a gauge field for it. By integrating out these gauge fields, we can obtain the T-dual background. All this work nicely because it is done in the bosonic string theory. In the superstring theory case, the situation is more involved because is difficult to work with Ramond backgrounds. This difficulty is avoided if we have a superstring theory formalism similar to the bosonic string. Such formalism is claimed to be the pure spinor formalism [3]. This covariant formalism for the superstring was invented some time ago by Berkovits. It uses the superspace coordinates as basic free world-sheet variables. Since this system is not conformal invariant, it is needed to introduce new bosonic variables λ^α constrained to satisfy $(\lambda\gamma^m\lambda) = 0$, where γ^m are the symmetric 16×16 gamma matrices in ten dimensions. These variable are named as pure spinors. Conformal invariance is not enough to quantize this sting theory. The new ingredient is to postulate the nilpotent charge $Q = \oint \lambda^\alpha d_\alpha$ (with d_α being the world-sheet generator of translations in superspace) as BRST charge of the system. Although it is necessary to break ten-dimensional Poincare invariance to solving the pure spinor constraint, it can be shown that the physical spectrum [4] and scattering amplitudes are manifestly super Poincare invariant [5].

Strings in curved backgrounds can be constructed in this formalism [6], where it was shown that BRST invariance implies that the background fields satisfy the corresponding ten-dimensional supergravity constraints. It was shown that this system preserves one-loop conformal invariance in the heterotic [7] and type II [8] strings as consequence of the classical BRST constraints found in [6]. It was also shown in [9] that the quantum BRST invariance is modified consistently after using cohomological methods[2].

In this note we will study the quantum preservation of the bosonic [11] and fermionic [12] T-dualities in the pure spinor formalism. Note that a combination of bosonic and fermionic T-dualities was used in [12] to show that the $AdS_5 \times S^5$ background of type IIB

[2] In [10], the gauge field contribution to the one-loop BRST invariance of the effective action was obtained.

string theory remains invariant explaining in this way the so called "dual superconformal symmetry" of certain planar scattering amplitudes in $N=4, d=4$ SYM theory [13] [14] (see also the review [15]).

In the next section we will review the bosonic T-duality of [11] and the fermionic duality of [12] for the heterotic superstring. Then we will explicitly check that the classical BRST constraints are preserved after a T-duality transformation. Finally we will prove that quantum conformal and quantum local symmetries are preserved under T-duality.

2. T-duality for the Heterotic Pure Spinor Superstring

We review the T-dualities discovered in [11] and [12] for the heterotic string case. The sigma model action in this case is given by

$$S = \frac{1}{2\pi\alpha'}\int d^2z\ \frac{1}{2}\partial Z^{\overline{M}}\overline{\partial} Z^{\overline{N}}(G_{\overline{NM}}(Z)+B_{\overline{NM}}(Z)) + \partial Z^{\overline{M}}\overline{J}^I A_{I\overline{M}}(Z) + d_\alpha \overline{\partial} Z^{\overline{M}} E_{\overline{M}}{}^{\alpha}(Z)$$
$$+d_\alpha \overline{J}^I W_I^\alpha(Z) + \lambda^\alpha\omega_\beta\overline{\partial}Z^{\overline{M}}\Omega_{\overline{M}\alpha}{}^{\beta}(Z) + \lambda^\alpha\omega_\beta\overline{J}^I U_{I\alpha}{}^{\beta}(Z) + \mathcal{L}(\overline{J},\lambda,\omega) + \mathcal{L}_{FT}, \tag{2.1}$$

where $Z^{\overline{M}}$ are the coordinates of the ten-dimensional heterotic superspace, $\overline{J}^I$ is the current for the heterotic fermions, d_α is the world-sheet generator for superspace translations, λ^α is the pure spinor and ω_α is its conjugate momentum. The term $\mathcal{L}(\overline{J},\lambda,\omega)$ is the action for the pure spinor variables and heterotic fermions in flat space. The term $\mathcal{L}_{FT}$ is the Fradkin-Tseytlin term and it is given by

$$\mathcal{L}_{FT} = \alpha'\mathcal{R}\Phi(Z), \tag{2.2}$$

where $\mathcal{R}$ is the two-dimensional scalar curvature and Φ is the dilaton superfield. The background fields $G, B, ...$ satisfy the supergravity equations of motion as consequence of the BRST invariance of (2.1). This symmetry is generated by the nilpotent charge $Q=\oint \lambda^\alpha d_\alpha$.

Now we will perform a combination of bosonic T-duality, as in [11], and then a fermionic T-duality, as in [12]. In both cases, we will see that the action (2.1) preserves its form.

2.1. Bosonic T-duality in the pure spinor string

We assume that the background fields are independent of some bosonic direction, say X^1, and we split $Z^{\overline{M}} = (X^1, Y^M)$. As it was noted in [2], the T-dual action is obtained after gauging the X^1-direction by introducing a purely gauge fields A and $\overline{A}$ as

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \frac{1}{2} A\overline{A} G_{11}(Y) + \frac{1}{2} A\overline{\partial} Y^M L_{M1}(Y) + \frac{1}{2} \partial Y^M \overline{A} L_{1M}(Y) + \frac{1}{2} \partial Y^M \overline{\partial} Y^N L_{NM}(Y) \tag{2.3}$$

$$+ A\overline{J}^I A_{I1}(Y) + \partial Y^M \overline{J}^I A_{IM}(Y) + d_\alpha \overline{A} E_1{}^\alpha(Y) + d_\alpha \overline{\partial} Y^M E_M{}^\alpha(Y) + d_\alpha \overline{J}^I W_I^\alpha(Y)$$

$$+ \lambda^\alpha \omega_\beta \overline{A} \Omega_{1\alpha}{}^\beta(Y) + \lambda^\alpha \omega_\beta \overline{\partial} Y^M \Omega_{M\alpha}{}^\beta(Y) + \lambda^\alpha \omega_\beta \overline{J}^I U_{I\alpha}{}^\beta(Y) + \frac{1}{2} \widetilde{X}^1 (\partial \overline{A} - \overline{\partial} A) + \mathcal{L}(\overline{J}, \lambda, \omega) + \mathcal{L}_{FT},$$

where $L_{\overline{NM}} = G_{\overline{NM}} + B_{\overline{NM}}$ and $\widetilde{X}^1$ is a lagrange multiplier which enforces the pure gauge condition. In fact, if we integrate out this field, we recover the original action (2.1) if

$$A = \partial X^1, \quad \overline{A} = \overline{\partial} X^1.$$

In order to obtain the T-dual sigma model action, we integrate out the gauge fields instead. By varying respect to A and $\overline{A}$, we obtain

$$A = (\partial \widetilde{X}^1 - \partial Y^M L_{1M} - 2 d_\alpha E_1{}^\alpha - 2\lambda^\alpha \omega_\beta \Omega_{1\alpha}{}^\beta) \frac{1}{G_{11}}, \tag{2.4}$$

$$\overline{A} = (-\overline{\partial} \widetilde{X}^1 - \overline{\partial} Y^M L_{M1} - 2\overline{J}^I A_{I1}) \frac{1}{G_{11}}.$$

We plug this in the action (2.3) to get

$$S = \frac{1}{2\pi\alpha'} \int d^2z \, \frac{1}{2} \partial \widetilde{X}^1 \overline{\partial} \widetilde{X}^1 G'_{11}(Y) + \frac{1}{2} \partial \widetilde{X}^1 \overline{\partial} Y^M L'_{M1}(Y) + \frac{1}{2} \partial Y^M \overline{\partial} \widetilde{X}^1 L'_{1M}(Y) \tag{2.5}$$

$$+ \frac{1}{2} \partial Y^M \overline{\partial} Y^N L'_{NM}(Y) + \partial \widetilde{X}^1 \overline{J}^I A'_{I1}(Y) + \partial Y^M \overline{J}^I A'_{IM}(Y) + d_\alpha \overline{\partial} \widetilde{X}^1 E'_1{}^\alpha(Y) + d_\alpha \overline{\partial} Y^M E'_M{}^\alpha(Y)$$

$$+ d_\alpha \overline{J}^I W'^\alpha_I(Y) + \lambda^\alpha \omega_\beta \overline{\partial} \widetilde{X}^1 \Omega'_{1\alpha}{}^\beta(Y) + \lambda^\alpha \omega_\beta \overline{\partial} Y^M \Omega'_{M\alpha}{}^\beta(Y)$$

$$+ \lambda^\alpha \omega_\beta \overline{J}^I U'_{I\alpha}{}^\beta(Y) + \mathcal{L}(\overline{J}, \lambda, \omega) + \mathcal{L}'_{FT},$$

where the transformed fields are given by

$$G'_{11} = \frac{1}{G_{11}}, \quad L'_{M1} = \frac{L_{M1}}{G_{11}}, \quad L'_{1M} = -\frac{L_{1M}}{G_{11}}, \quad L'_{NM} = L_{NM} - \frac{L_{N1}L_{1M}}{G_{11}}, \tag{2.6}$$

$$A'_{I1} = \frac{A_{I1}}{G_{11}}, \quad A'_{IM} = A_{IM} - \frac{A_{I1}L_{1M}}{G_{11}}, \quad E'_1{}^{\alpha} = -\frac{E_1{}^{\alpha}}{G_{11}}, \quad E'_M{}^{\alpha} = E_M{}^{\alpha} - \frac{L_{M1}E_1{}^{\alpha}}{G_{11}},$$

$$\Omega'_{1\alpha}{}^{\beta} = -\frac{\Omega_{1\alpha}{}^{\beta}}{G_{11}}, \quad \Omega'_{M\alpha}{}^{\beta} = \Omega_{M\alpha}{}^{\beta} - \frac{L_{M1}\Omega_{1\alpha}{}^{\beta}}{G_{11}},$$

$$W'^{\alpha}_I = W^{\alpha}_I - 2\frac{A_{I1}E_1{}^{\alpha}}{G_{11}}, \quad U'_{I\alpha}{}^{\beta} = U_{I\alpha}{}^{\beta} - 2\frac{A_{I1}\Omega_{1\alpha}{}^{\beta}}{G_{11}}.$$

It remains to determine the bosonic component of the supervielbein $E_{\overline{M}}{}^{a}$. It can be obtained through the definition

$$G_{\overline{NM}} = E_{\overline{N}}{}^{a} E_{\overline{M}}{}^{b} \eta_{ab}. \tag{2.7}$$

Following [11], we determine this supervielbein as

$$E'_{\overline{M}}{}^{a} = Q_{\overline{M}}{}^{\overline{N}} E_{\overline{N}}{}^{a}, \tag{2.8}$$

where the matrix Q is determined by

$$E'_{\overline{M}}{}^{\alpha} = Q_{\overline{M}}{}^{\overline{N}} E_{\overline{N}}{}^{\alpha}. \tag{2.9}$$

According to (2.6), the matrix Q has the entries

$$Q_1{}^1 = -\frac{1}{G_{11}}, \quad Q_1{}^M = 0, \quad Q_M{}^1 = -\frac{L_{M1}}{G_{11}}, \quad Q_N{}^M = \delta_M{}^N. \tag{2.10}$$

In this way we obtain

$$E'_1{}^{a} = -\frac{E_1{}^{a}}{G_{11}}, \quad E'_M{}^{a} = E_M{}^{a} - \frac{L_{M1}E_1{}^{\alpha}}{G_{11}}. \tag{2.11}$$

As verification, we need to obtain the transformations given in (2.6) for the supermetric by using the definition of (2.7) and (2.11). In fact, doing this

$$G'_{11} = \frac{1}{G_{11}}, \quad G'_{1M} = \frac{B_{M1}}{G_{11}}, \quad G'_{NM} = G_{NM} + \frac{B_{N1}B_{M1} - G_{N1}G_{M1}}{G_{11}}.$$

which are compatible with (2.6). Note that we need to perform a shift in de dilaton superfield too because of the term involving $A\overline{A}$ in (2.3) is not one. As it was shown in [2], the dilaton is transformed according to

$$\Phi' = \Phi + \log \frac{2\pi\alpha'}{G_{11}}. \tag{2.12}$$

We will also need the inverse of the super vielbein. It was found that $E'_{\overline{M}}{}^{A} = Q_{\overline{M}}{}^{\overline{N}} E_{\overline{N}}{}^{A}$ where the matrix Q is given in (2.10). The inverse $E'_{A}{}^{\overline{M}} = E_{A}{}^{N} (Q^{-1})_{\overline{N}}{}^{\overline{M}}$, where the entries for the matrix Q^{-1} are

$$(Q^{-1})_1{}^1 = -G_{11}, \quad (Q^{-1})_1{}^M = 0, \quad (Q^{-1})_M{}^1 = -L_{M1}, \quad (Q^{-1})_M{}^N = \delta_M{}^N. \tag{2.13}$$

2.2. Fermionic T-duality

Let us consider now a fermionic T-duality [12]. We assume that the background in the sigma model action (2.1) is independent of a fermionic direction, say θ^1. As in the bosonic case, we gauge this isometry by introducing a pair of fermionic gauge field $(A, \overline{A})$ and by adding a fermionic Lagrange multiplier which enforces a pure gauge condition on the gauge fields. The action now is

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2 z\, & \frac{1}{2} A\overline{A} B_{11}(Y) + \frac{1}{2} A\overline{\partial} Y^M L_{M1}(Y) + \frac{1}{2} \partial Y^M \overline{A} L_{1M}(Y) + \frac{1}{2} \partial Y^M \overline{\partial} Y^N L_{NM}(Y) \\ & + A\overline{J}^I A_{I1}(Y) + \partial Y^M \overline{J}^I A_{IM}(Y) + d_\alpha \overline{A} E_1{}^\alpha(Y) + d_\alpha \overline{\partial} Y^M E_M{}^\alpha(Y) + d_\alpha \overline{J}^I W_I^\alpha(Y) \\ & + \lambda^\alpha \omega_\beta \overline{A} \Omega_{1\alpha}{}^\beta(Y) + \lambda^\alpha \omega_\beta \overline{\partial} Y^M \Omega_{M\alpha}{}^\beta(Y) + \lambda^\alpha \omega_\beta \overline{J}^I U_{I\alpha}{}^\beta(Y) + \frac{1}{2} \widetilde{\theta}^1 (\partial \overline{A} - \overline{\partial} A) + \mathcal{L}(\overline{J}, \lambda, \omega) + \mathcal{L}_{FT}. \end{aligned} \tag{2.14}$$

Integrating out the fermionic Lagrange multiplier $\widetilde{\theta}^1$, we obtain the action (2.1) because

$$A = \partial\theta^1, \quad \overline{A} = \overline{\partial}\theta^1.$$

We now integrate the gauge fields. The equations of motion for them determine

$$A = (\partial\widetilde{\theta}^1 - (-1)^M \partial Y^M L_{1M} - 2 d_\alpha E_1{}^\alpha + 2\lambda^\alpha \omega_\beta \Omega_{1\alpha}{}^\beta) \frac{1}{B_{11}}, \tag{2.15}$$

$$\overline{A} = (\overline{\partial}\widetilde{\theta}^1 - \overline{\partial} Y^M L_{M1} - 2\overline{J}^I A_{I1}) \frac{1}{B_{11}},$$

where $(-1)^M$ is -1 if M is a bosonic index and is $+1$ if M is a fermionic index. We plug these values in the action (2.14) to obtain the fermionic T-dual background

$$S = \frac{1}{2\pi\alpha'}\int d^2z\ \frac{1}{2}\partial\widetilde{\theta}^1\overline{\partial}\widetilde{\theta}^1 B'_{11}(Y) + \frac{1}{2}\partial\widetilde{\theta}^1\overline{\partial}Y^M L'_{M1}(Y) + \frac{1}{2}\partial Y^M\overline{\partial}\widetilde{\theta}^1 L'_{1M}(Y) \tag{2.16}$$

$$+\frac{1}{2}\partial Y^M\overline{\partial}Y^N L'_{NM}(Y)+\partial\widetilde{\theta}^1\overline{J}^I A'_{I1}(Y)+\partial Y^M\overline{J}^I A'_{IM}(Y)+d_\alpha\overline{\partial}\widetilde{\theta}^1 E_1'^{\ \alpha}(Y)+d_\alpha\overline{\partial}Y^M E_M'^{\ \alpha}(Y)$$

$$+d_\alpha\overline{J}^I W'^{\alpha}_I(Y) + \lambda^\alpha\omega_\beta\overline{\partial}\widetilde{\theta}^1\Omega'_{1\alpha}{}^\beta(Y) + \lambda^\alpha\omega_\beta\overline{\partial}\widetilde{\theta}^1\Omega'_{1\alpha}{}^\beta(Y) + \lambda^\alpha\omega_\beta\overline{\partial}Y^M\Omega'_{M\alpha}{}^\beta(Y)$$

$$+\lambda^\alpha\omega_\beta\overline{J}^I U'_{I\alpha}{}^\beta(Y) + \mathcal{L}(\overline{J},\lambda,\omega) + \mathcal{L}'_{FT},$$

where the fermionic T-dual background is given by

$$B'_{11} = -\frac{1}{B_{11}},\quad L'_{M1} = \frac{L_{M1}}{B_{11}},\quad L'_{1M} = \frac{L_{1M}}{B_{11}},\quad L'_{NM} = L_{NM} - \frac{L_{N1}L_{1M}}{B_{11}}, \tag{2.17}$$

$$A'_{I1} = \frac{A_{I1}}{B_{11}},\quad A'_{IM} = A_{IM} - \frac{A_{I1}L_{1M}}{B_{11}},\quad E_1'^{\ \alpha} = \frac{E_1{}^\alpha}{B_{11}},\quad E_M'^{\ \alpha} = E_M{}^\alpha - \frac{L_{M1}E_1{}^\alpha}{B_{11}},$$

$$\Omega'_{1\alpha}{}^\beta = \frac{\Omega_{1\alpha}{}^\beta}{B_{11}},\quad \Omega'_{M\alpha}{}^\beta = \Omega_{M\alpha}{}^\beta - \frac{L_{M1}\Omega_{1\alpha}{}^\beta}{B_{11}},$$

$$W'^{\alpha}_I = W^\alpha_I - 2\frac{A_{I1}E_1{}^\alpha}{B_{11}},\quad U'_{I\alpha\beta} = U_{I\alpha}{}^\beta - 2\frac{A_{I1}\Omega_{1\alpha}{}^\beta}{B_{11}}.$$

We note that these transformations are very similar to those in the bosonic case. In the fermionic case, the B_{11} plays the role of G_{11} in the bosonic case. It remains to determine the bosonic component of the supervielbein. As in the bosonic case, we note that is given by (2.8) where the matrix Q is determined by (2.9) and the transformations of (2.17). The entries of Q are

$$Q_1{}^1 = \frac{1}{B_{11}},\quad Q_1{}^M = 0,\quad Q_M{}^1 = -\frac{L_{M1}}{B_{11}},\quad Q_N{}^M = \delta_M{}^N. \tag{2.18}$$

Therefore,

$$E_1'^{\ a} = \frac{E_1{}^a}{B_{11}},\quad E_M'^{\ a} = E_M{}^a - \frac{L_{M1}E_1{}^\alpha}{B_{11}}. \tag{2.19}$$

From the definition (2.7), we determine that the transformations of the supersymmetric components are

$$G'_{M1} = \frac{G_{M1}}{B_{11}}, \quad G'_{NM} = G_{NM} - \frac{G_{N1}B_{1M} + B_{N1}G_{1M}}{B_{11}}.$$

which are compatible with (2.17). As before, the dilaton superfield is shifted as

$$\Phi' = \Phi - \log \frac{2\pi\alpha'}{B_{11}}. \tag{2.20}$$

Here the relative sign is reverted respect to (2.12) because the measure for the gauge field is grassmannian, then the jacobian of the transformation is inverted [12].

3. Classical BRST Invariance

In the previous section it was shown how the T-dualities are realized in the pure spinor heterotic string. The classical BRST invariance of the sigma model action puts them background field on-shell [6]. Since the BRST charge $Q = \oint \lambda^\alpha d_\alpha$ is unaffected by the T-duality transformations, then the T-dual background is also a solution of the $N = 1$ ten-dimensional supergravity/SYM equations of motion. Therefore, we expect that the T-dual sigma model action, both bosonic and fermionic, is also BRST invariant. Now it will be shown that this is the case.

The classical BRST invariance of the action implies that the background fields satisfy the constraints [6] (see also [16][3]

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^A = \lambda^\alpha \lambda^\beta H_{\alpha\beta A} = \lambda^\alpha \lambda^\beta F_{I\alpha\beta} = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = 0. \tag{3.1}$$

Now it will be shown that these constraints remain after a T-dual transformation of the background. Consider the bosonic T-duality of (2.6), the fermionic case is analogous.

Let's start with $\lambda^\alpha \lambda^\beta F_{I\alpha\beta} = 0$. We use that

$$F'_{I\alpha\beta} = (-1)^{\overline{M}+1} E'^{\;\overline{M}}_{\beta} E'^{\;\overline{N}}_{\alpha} F'_{I\overline{N}\overline{M}}.$$

Now we split the index $\overline{M}$ into $(1, M)$ and we use the definition of F in terms of the gauge potential A together with the transformations of (2.6) to obtain

$$F'_{I\alpha\beta} = F_{I\alpha\beta} - \frac{1}{G_{11}}(H_{\alpha\beta 1} + T_{\alpha\beta}{}^a E_1{}^b \eta_{ab}) A_{I1}.$$

[3] The constraints coming from the holomorphicity of the BRST current are implied by these and Bianchi identities.

If we hit this expression with $\lambda^\alpha\lambda^\beta$ we obtain

$$\lambda^\alpha\lambda^\beta F'_{I\alpha\beta} = \lambda^\alpha\lambda^\beta F_{I\alpha\beta} = 0.$$

Consider now the constraint involving H. Starting from

$$H'_{\alpha\beta A} = (-1)^{\overline{M}+1} E'_A{}^{\overline{P}} E'_\beta{}^{\overline{N}} E'_\alpha{}^{\overline{M}} H'_{\overline{MNP}},$$

by splitting the index $\overline{M}$ as $(1, M)$, the definition of H in terms of B and the T-dual transformations (2.6) we obtain

$$H'_{\alpha\beta A} = H_{\alpha\beta A} + \frac{1}{G_{11}} E_\alpha{}^{\overline{M}} G_{\overline{M}1} E_1{}^a T_{\beta A}{}^b \eta_{ab} = H_{\alpha\beta A}.$$

Then, the constraint $\lambda^\alpha\lambda^\beta H_{\alpha\beta A} = 0$ is preserved.

Consider the torsion constraint in (3.1). Starting from

$$T'_{\alpha\beta}{}^A = (-1)^{\overline{M}+1} E'_\beta{}^{\overline{M}} E'_\alpha{}^{\overline{N}} T'_{\overline{NM}}{}^A,$$

using the definition of the torsion as the covariant derivative of the vielbein and splitting index $\overline{M}$ into $(1, M)$ we obtain

$$T'_{\alpha\beta}{}^A = T_{\alpha\beta}{}^A - \frac{1}{G_{11}} (H_{\alpha\beta 1} + T_{\alpha\beta}{}^a E_1{}^b \eta_{ab}) E_1{}^A,$$

from which we obtain

$$\lambda^\alpha\lambda^\beta T'_{\alpha\beta}{}^A = \lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^A.$$

Finally, consider the torsion constraint in (3.1). Starting from

$$R'_{\alpha\beta\gamma}{}^\delta = (-1)^{\overline{M}+1} E'_\beta{}^{\overline{M}} E'_\alpha{}^{\overline{N}} R'_{\overline{NM}\gamma}{}^\delta,$$

using the definition of R in terms of the connection Ω and splitting the index $\overline{M}$ into $(1, M)$ we obtain

$$R'_{\alpha\beta\gamma}{}^\delta = R_{\alpha\beta\gamma}{}^\delta - \frac{\Omega_{1\gamma}{}^\delta}{G_{11}} (H_{\alpha\beta 1} + T_{\alpha\beta}{}^a E_1{}^b \eta_{ab}),$$

then, we get

$$\lambda^\alpha\lambda^\beta R'_{\alpha\beta\gamma}{}^\delta = \lambda^\alpha\lambda^\beta R_{\alpha\beta\gamma}{}^\delta.$$

4. Quantum super T-duality

In this section we discuss the preservation of some symmetries of the sigma model action (2.1) under background T-duality transformations. We consider the conformal, gauge and local Lorentz symmetries

4.1. One-loop conformal invariance

As we have already stated, the classical BRST invariance of the sigma model action puts the background fields on-shell [6]Ṡince the BRST charge $Q = \oint \lambda^\alpha d_\alpha$ is unaffected by the T-duality transformations, then the T-dual background is also a solution of the $N = 1$ ten-dimensional supergravity/SYM equations of motion. As it was shown in [7], the one-loop conformal invariance of the sigma model action is a consequence of the classical BRST invariance. Therefore, we expect that the T-dual sigma model action, both bosonic and fermionic, is also BRST invariant. Now it will be shown that this is the case.

The classical BRST invariance of the action implies that the background fields satisfy the constraints (3.1). Recall that the Ω connection here involves the usual Lorentz connection and a connection for the scaling invariance of (2.1) through

$$\Omega_{M\alpha}{}^{\beta} = \Omega_M \delta_\alpha{}^{\beta} + \frac{1}{4}(\gamma^{ab})_\alpha{}^{\beta}\Omega_{Mab}. \tag{4.1}$$

After performing a covariant background field expansion, it was shown in [7] that the one-loop UV divergence of the effective action vanishes after using the constraints of (3.1), Bianchi identities and the relation

$$\nabla_\alpha \Phi = 4\ \Omega_\alpha, \tag{4.2}$$

where Φ is a superfield which appears in the Fradkin-Tseytlin term in (2.2) and $\Omega_\alpha = E_\alpha{}^M \Omega_M$. The relation (4.2) can be obtained by requiring the vanishing of the ghost number anomaly as it was discussed in [6] and [8].

In the T-dual background, the BRST constraints are equivalent to (3.1) but with the curvatures constructed the T-dual connections of (2.6) in the bosonic case or the connections of (2.17) in the fermionic case. Now one can perform a covariant background field expansion and demonstrate the vanishing of the one-loop effective action. In order to do this, it is necessary to use the relation (4.1) in the T-dual background, that is

$$\nabla'_\alpha \Phi' = 4\ \Omega'_\alpha, \tag{4.3}$$

where the covariant derivative is defined with the T-dual transformed connections of (2.6) or (2.17), and Φ' is given by (2.12) or (2.20). There is an apparent contradiction because if one uses the rules for transforming the background field under a T-dual transformation, the rhs of (4.3) remains invariant. Let us prove this assertion in the bosonic case. In the fermionic case, the proof can be done in parallel.

Using this,

$$\Omega'_\alpha = E'_\alpha{}^{\overline{M}}\Omega'_{\overline{M}} = -G_{11}E_\alpha{}^1\Omega'_1 - E_\alpha{}^N L_{N1}\Omega'_1 + E_\alpha{}^M\Omega'_M = \Omega_\alpha.$$

Analogously, the lhs of (4.3) transforms as

$$\nabla'_\alpha\Phi' = \nabla_\alpha\Phi - \frac{1}{G_{11}}\nabla_\alpha G_{11}.$$

By combining both transformations, we find a contradiction with (4.2). This is solved by recalling that the action of (2.1) is invariant under a scaling transformations where $\delta\lambda^\alpha = \epsilon\lambda^\alpha, \delta\omega_\alpha = -\epsilon\omega_\alpha, \ldots$ and $\delta\Omega_M = -\partial_M\epsilon$. Using this symmetry, we just change $\Omega_M \to \Omega_M - \frac{1}{4}\nabla_M \log G_{11}$ to preserve the equation (4.2).

4.2. Gauge and local Lorentz symmetries

The action of the pure spinor heterotic string (2.1) is invariant under local gauge and local Lorentz transformations. Under the former, the background field $A_{I\overline{M}}$ transforms as a connection and all other fields carrying an index I transform in the adjoint representation of the gauge group. Of course, fields without an index I are inert under the gauge group. Under a local Lorentz transformation, Ω is the connection and all other fields transform homogenously.

It is well known that the effective action is potentially anomalous because these symmetries act on chiral fermions. Consider first the heterotic fermions in the action of (2.1). The effective action is determined by performing a covariant background expansion and then integrating out the quantum fluctuations (see [17] and references therein). It turns out the part of the effective action involving the heterotic fermions is given by

$$e^{-S_{eff}[a]} = \int D\rho\; e^{\frac{1}{4\pi\alpha'}\int d^2z\ Tr(\rho\nabla\rho)}, \tag{4.4}$$

where the trace is over the vector representation of the gauge group, the heterotic fermions belong to this representation, the covariant derivative is given by $\nabla\rho = \partial\rho + a\rho$, where $a = a_I K^I$ with K^I denoting the generators of the gauge group and

$$a_I = \partial Z^{\overline{M}} A_{I\overline{M}} + d_\alpha W_I^\alpha + \lambda^\alpha \omega_\beta U_{I\alpha}{}^\beta, \tag{4.5}$$

which is determined in the covariant background field expansion performed in [7]. The gauge anomaly is given by making a gauge transformation with the gauge field in the combination of (4.5). Now it will be shown that this world-sheet field is invariant under the bosonic T-duality (2.6). In fact,

$$a_I' = \partial \widetilde{X}^1 A_{I1}' + \partial Y^M A_{IM}' + d_\alpha W_I'^\alpha + \lambda^\alpha \omega_\beta U_{I\alpha}'{}^\beta = \partial \widetilde{X}^1 \frac{A_{I1}}{G_{11}} + \partial Y^M (A_{IM} - \frac{A_{I1} L_{1M}}{G_{11}})$$

$$+ d_\alpha (W_I^\alpha - 2\frac{A_{I1} E_1{}^\alpha}{G_{11}}) + \lambda^\alpha \omega_\beta (U_{I\alpha}{}^\beta - 2\frac{A_{I1} \Omega_{1\alpha}{}^\beta}{G_{11}}),$$

by using the equations (2.4) and that $A = \partial X^1, \overline{A} = \overline{\partial} X^1$, we obtain that $a_I' = a_I$. Therefore, the anomalies determined by this world-sheet field are invariant under the bosonic T-duality of (2.6).

Consider now the fermionic T-duality case. We obtain

$$a_I' = \partial \widetilde{\theta}^1 A_{I1}' + \partial Y^M A_{IM}' + d_\alpha W_I'^\alpha + \lambda^\alpha \omega_\beta U_{I\alpha}'{}^\beta = \partial \widetilde{\theta}^1 \frac{A_{I1}}{B_{11}} + \partial Y^M (A_{IM} - \frac{A_{I1} L_{1M}}{B_{11}})$$

$$+ d_\alpha (W_I^\alpha - 2\frac{A_{I1} E_1 \alpha}{B_{11}}) + \lambda^\alpha \omega_\beta (U_{I\alpha}{}^\beta - 2\frac{A_{I1} \Omega_{1\alpha}{}^\beta}{B_{11}}),$$

by using the equations (2.15) and that $A = \partial \theta^1, \overline{A} = \overline{\partial} \theta^1$, we obtain that $a_I' = a_I$. Therefore, we have verified that the T-dualities do not affect the gauge anomaly in the heterotic string case

Now we consider the local Lorentz symmetry of the action (2.1). We have two potentially anomalous chiral systems, one coming from the expansion of $(d_\alpha, Z^{\overline{M}} E_{\overline{M}}{}^\beta)$ and the other from the expansion of the pure spinor ghosts $(\omega_\alpha, \lambda^\beta)$. In both cases, the effective action is of the form

$$e^{-S_{eff}[\overline{\Delta}]} = \int D\psi D\varphi \; e^{-\frac{1}{2\pi\alpha'} \int d^2 z \;\; \psi_\alpha \overline{\nabla} \varphi^\alpha}, \tag{4.6}$$

where $(\varphi_\alpha, \psi_\beta)$ is $(Z^{\overline{M}} E_{\overline{M}}{}^\alpha, d_\beta)$ or $(\lambda^\alpha, \omega_\beta)$ and the covariant derivative is given by $\overline{\nabla} \varphi^\alpha = \overline{\partial} \varphi^\alpha + \varphi^\beta \overline{\Delta}_\beta{}^\alpha$ with the connection given by

$$\overline{\Delta}_\alpha{}^\beta = \overline{\partial} Z^{\overline{M}} \Omega_{\overline{M}\alpha}{}^\beta + \overline{J}^I U_{I\alpha}{}^\beta. \tag{4.7}$$

Now it will be shown that (4.7) is invariant under T-dualities. Using the rules of (2.6) for the bosonic T-duality we obtain

$$\overline{\Delta}'_\alpha{}^\beta = \overline{\partial}\widetilde{X}^1 \Omega'_{1\alpha}{}^\beta + \overline{\partial} Y^M \Omega'_{M\alpha}{}^\beta + \overline{J}^I U'_{I\alpha}{}^\beta = -\overline{\partial}\widetilde{X}^1 \frac{\Omega_{1\alpha}{}^\beta}{G_{11}} + \overline{\partial} Y^M (\Omega_{M\alpha}{}^\beta - \frac{L_{M1}\Omega_{1\alpha}{}^\beta}{G_{11}})$$

$$+\overline{J}^I (U_{I\alpha}{}^\beta - 2\frac{A_{I1}\Omega_{1\alpha}{}^\beta}{G_{11}}) = \overline{\Delta}_\alpha{}^\beta,$$

where we have used the equations of (2.4) and $\overline{A} = \overline{\partial} X^1$.

Analogously, the fermionic T-duality also leaves invariant the world-sheet field $\overline{\Delta}$. Then, we can conclude that the anomalies which depend on $\overline{\Delta}$, as the local Lorentz anomaly, will not be affected by T-dualities.

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