

Conserved charges for even dimensional asymptotically AdS gravity theories

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Mass and other conserved Noether charges are discussed for solutions of gravity theories with locally anti-de Sitter (AdS) asymptotics in $2n$ dimensions. The action is supplemented with a boundary term whose purpose is to guarantee that it reaches an extremum on the classical solutions, provided the space-time is locally AdS space-time at the boundary. It is also shown that if space-time is locally AdS at spatial infinity, the conserved charges are finite and properly normalized without requiring subtraction of a reference background. In this approach, Noether charges associated with Lorentz and diffeomorphism invariance vanish identically for constant curvature space-times. The case of a zero cosmological constant is obtained as a limit of AdS space-time, where Λ plays the role of a regulator.

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I. INTRODUCTION

Noether's theorem is the standard tool in theoretical physics to construct conserved charges associated with invariances of the action. Nevertheless, general relativity, described by the Einstein-Hilbert action, does not lend itself naturally to the application of Noether's theorem. The conserved charge associated with the invariance of the action under diffeomorphisms is given by Komar's formula [1]

$$K(\xi) = -\kappa \int_{\partial\Sigma} \nabla^\mu \xi^\nu d\Sigma_{\mu\nu}, \quad (1)$$

where $\kappa = (16\pi G)^{-1}$, $\xi = \xi^\mu \partial_\mu$ is a vector field that defines the diffeomorphism, ∇_μ represents the covariant derivative in terms of Christoffel symbol, $\partial\Sigma$ is the boundary of the spatial section, and $d\Sigma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta$ is the surface element (dual of the area two-form). Then, when ξ is a time-like or rotational Killing vector, $K(\xi)$ provides a definition of mass or angular momentum, respectively. However, there

is a first drawback in this result, that is, in the case of the (3+1)-dimensional Kerr black hole, Eq. (1) gives the following answer:

$$K\left(\frac{\partial}{\partial t}\right) = \frac{M}{2}, \quad K\left(\frac{\partial}{\partial \phi}\right) = J. \quad (2)$$

These results show that there is no common normalization factor which could give the correct values for mass and angular momentum. Moreover, there is a second drawback with Komar's formula: in the presence of negative cosmological constant, spacetime is no longer asymptotically flat and the formula yields a divergent value. For example, for the Schwarzschild-anti-de Sitter (AdS) metric, one obtains

$$K\left(\frac{\partial}{\partial t}\right) = \frac{M}{2} + \lim_{r \rightarrow \infty} \frac{r^3}{2l^2}. \quad (3)$$

The standard approach to deal with this divergence is to subtract the value of $K(\xi)$ on the AdS background from Eq. (3) (see, e.g., Ref. [2]). In spite of giving a finite result, this does not correct the normalization factor of M and the first problem mentioned above remains.

The usual procedure to evaluate the conserved charges is the Arnowitt-Deser-Misner (ADM) formalism [3], which yields the correct formulas for the energy-momentum and angular momentum for asymptotically flat spacetimes. Nevertheless, this approach and its further extension developed

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by Regge and Teitelboim [4] provides a formula for the variation of the charges, e.g., δM , and in order to evaluate the charges, e.g., M , it is necessary to fix the reference background geometry. The Hamiltonian method can also be extended to provide the correct mass and angular momentum for asymptotically AdS spacetimes representing solutions of the Einstein-Hilbert action with negative cosmological constant in $d=4$ [5], as well as for $d \neq 4$ [6].

In many instances this scheme is sufficiently satisfactory, but there are some cases of physical interest in which the asymptotic behavior can be difficult to assess, as in the case of asymptotically *locally* anti-de Sitter (ALAdS) spaces. A formalism to define ‘‘conserved’’ charges in asymptotically AdS spaces was proposed by Ashtekar and Magnon [7], who used conformal techniques to construct the conserved quantities. This construction makes no reference to an action, and yet reproduces the charges obtained by Hamiltonian methods [5].

Another scheme has been recently proposed by Balasubramanian and Kraus [8] who use the Einstein-Hilbert action with Dirichlet boundary conditions for the metric, supplemented by counterterms in order to ensure the finiteness of the stress tensor derived by the quasilocal energy definition [9]. By adding a finite series of local invariants of the boundary geometry, the counterterm action regularizes that definition of energy. This idea was subsequently extended to higher dimensions in Ref. [10]. A different, non-polynomial, expression has been given by Ref. [11], which reduces to the previous one in the infinite cosmological constant limit. In Ref. [12], an alternative construction is proposed which yields the conserved charges in (3+1)-dimensional general relativity with negative cosmological constant and does not need to specify the background provided it is ALAdS.

Noether charges in 3+1 ALAdS gravity

The approach presented in Ref. [12] leads to a properly defined, convergent expression for the Noether charges in 3+1 dimensions, provided the ALAdS boundary condition is imposed on the manifold. It is important to note that the local AdS behavior at the boundary is not equivalent to the usual Dirichlet condition over the metric, in order to have a well defined variational principle.

The situation in 3+1 dimensions is reviewed in order to set the basic facts in the construction. The starting observation is that the ALAdS condition requires adding a boundary term to the Einstein-Hilbert action equal to the Euler density (Gauss-Bonnet term) with a fixed weight factor, in order to cancel the boundary term coming from the variation of the Lagrangian. As a consequence, the action, including the boundary term, is

$$I = \frac{\kappa l^2}{4} \int_{\mathcal{M}} \epsilon_{abcd} \bar{R}^{ab} \bar{R}^{cd}, \quad (4)$$

where $\bar{R}^{ab} := R^{ab} + l^{-2} e^a e^b$. The Noether charge computed with the action (4) has the right normalization factor and is

finite for (3+1)-dimensional ALAdS spaces. This charge, associated with the invariance under a diffeomorphism of Eq. (4), is

$$Q(\xi) = \frac{\kappa l^2}{2} \int_{\partial\Sigma} \epsilon_{abcd} I_\xi \omega^{ab} \bar{R}^{cd}, \quad (5)$$

where $\xi^\mu = x'^\mu - x^\mu$ is the arbitrary vector field¹ that generates the diffeomorphism.

Although Komar’s formula and Eq. (5) are obtained as the conserved Noether charge associated with the same invariance, they disagree because the starting Lagrangians differ by a closed form and are deduced using second and first order formalism, respectively. In order to clarify this point, it is useful to split the charge (5) in such a way that the relation with the usual tensor formalism becomes explicit. $Q(\xi)$ can be written as

$$Q(\xi) = K(\xi) + X(\xi) + \frac{\kappa l^2}{2} \int_{\partial\Sigma} \epsilon_{abcd} I_\xi \omega^{ab} R^{cd}, \quad (6)$$

where $K(\xi)$ is given by Eq. (1), $X(\xi)$ is a contribution due to the local Lorentz invariance²

$$X(\xi) = -\frac{\kappa}{2} \int_{\partial\Sigma} \epsilon_{abcd} \Phi^{ab} e^c e^d, \quad (7)$$

with $\Phi^{ab} = e^{a\mu} \mathcal{L}_\xi e_\mu^b$ and the last term arises from the surface term in the action (which was set as $\kappa l^2/4$ times Euler density). When ξ is a Killing vector, Φ^{ab} can be shown to be antisymmetric and be identified as a local Lorentz transformation. In the second order formalism (7) is absent since there is no local Lorentz invariance.³

The last term in Eq. (6) plays a double role: it cancels the divergences which appear in the explicit evaluation of the solutions and contributes to the right normalization factor as well. In this sense, this term regularizes the Noether charge for ALAdS spaces. This can be checked explicitly in the following example: Consider the Schwarzschild-AdS solution and $\xi = \partial_t$. In the standard frame choice Φ^{ab} is zero and hence $X(\xi)$ vanishes. Evaluating Eq. (6) yields

$$K(\xi) = \frac{M}{2} + \lim_{r \rightarrow \infty} \frac{r^3}{2l^2}, \quad (8)$$

$$\frac{\kappa l^2}{2} \int_{\partial\Sigma} \epsilon_{abcd} I_\xi \omega^{ab} R^{cd} = \frac{M}{2} - \lim_{r \rightarrow \infty} \frac{r^3}{2l^2}, \quad (9)$$

and hence, $Q(\xi) = M$.

¹The action of the contraction operator I_ξ over a p -form $\alpha_p = (1/p!) \alpha_{\mu^1 \dots \mu^p} dx^{\mu^1} \dots dx^{\mu^p}$ is given by $I_\xi \alpha_p := 1/(p-1)! \xi^\nu \alpha_{\nu \mu^1 \dots \mu^{p-1}} dx^{\mu^1} \dots dx^{\mu^{p-1}}$. In terms of this operator, the Lie derivative reads $\mathcal{L}_\xi = dI_\xi + I_\xi d$.

²Here, the identity $\mathcal{L}_\xi e^a = D\xi^a - I_\xi \omega_b^a e^b$, which holds in Riemannian (torsion-free) manifolds has been used to obtain Eq. (6).

³Although it is always possible to choose a frame (e^a) such that $\Phi^{ab} = 0$ in an open neighborhood, there could be interesting cases where a global obstruction makes $X(\xi)$ nontrivial.

It is apparent from relations (8), (9), that the result (6) remains unchanged if the limit $l \rightarrow \infty$ is taken at the end. This permits applying the formula equally well for all values of the cosmological constant, including $\Lambda = 0$. In this sense, Λ can be regarded as a regulator for general relativity in the absence of cosmological constant.

In what follows, the extension of this approach to $2n$ -dimensional gravity theories is presented. Since in higher dimensions, the Einstein-Hilbert (EH) action is not the only option (see Sec. III below), we will also consider a particular extension of the so-called Lanczos-Lovelock actions, which has been dubbed the Born-Infeld (BI) action [13]. This is an example that the formalism can be applied to other theories of gravity that include higher powers of curvature R^{ab} .

II. EINSTEIN-HILBERT ACTION

A. Action principle

In this section a well defined first order action principle for EH Lagrangian in even dimensional ALAdS spacetimes is proposed. As in 3+1 dimensions, the existence of an extremum for ALAdS spaces fixes the boundary term that must be added to the action as proportional to the Euler density. Applying of Noether's theorem to this action yields a regularized, background-independent expression for the conserved charges.

The action to be considered is

$$I = I_{\text{EH}} + B, \quad (10)$$

where I_{EH} is the standard Einstein-Hilbert action with negative cosmological constant in $d = 2n$ dimensions

$$I_{\text{EH}} = \frac{\kappa}{2(n-1)} \int_{\mathcal{M}} \epsilon_{a_1 \dots a_d} \left(R^{a_1 a_2} e^{a_3} \dots e^{a_d} + \frac{d-2}{l^2 d} e^{a_1} \dots e^{a_d} \right), \quad (11)$$

and B is a boundary term.⁴

The on-shell variation of the action yields the boundary term

$$\delta I = \int_{\partial \mathcal{M}} \Theta, \quad (12)$$

where

$$\int_{\partial \mathcal{M}} \Theta = \frac{\kappa}{(d-2)} \int_{\partial \mathcal{M}} \epsilon_{a_1 \dots a_d} \delta \omega^{a_1 a_2} e^{a_3} \dots e^{a_d} + \delta B. \quad (13)$$

⁴Here, wedge product \wedge between differential forms is understood. The gravitational constant has been chosen as $\kappa = [2(d-2)! \Omega_{d-2}]^{-1}$ with Ω_{d-2} the volume of S^{d-2} .

Therefore, the action becomes stationary demanding $\Theta = 0$. Assuming the spacetime to be ALAdS ($\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b = 0$), the vanishing of Eq. (13) term is satisfied if

$$\delta B = n \alpha_n \int_{\partial \mathcal{M}} \epsilon_{a_1 \dots a_{2n}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}}, \quad (14)$$

where α_n is defined as

$$\alpha_n = \kappa \frac{(-1)^n l^{2n-2}}{2n(n-1)}. \quad (15)$$

The right-hand side (RHS) of (14) can be recognized as the variation of the $2n$ -dimensional Euler density⁵

$$\delta \mathcal{E}_{2n} = n \int_{\mathcal{M}} d[\epsilon_{a_1 \dots a_{2n}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}}].$$

Thus, the boundary term in Eq. (10) reads

$$B = \alpha_n \int_{\mathcal{M}} \mathcal{E}_{2n},$$

and the final expression for the action supplemented by the boundary term is

$$I = I_{\text{EH}} + \alpha_n \int_{\mathcal{M}} \mathcal{E}_{2n}. \quad (16)$$

This particular form of action is our starting point for the construction of the conserved charges. The topology of the manifold is assumed to be $\mathcal{M} = R \times \Sigma$.

The diffeomorphism invariance is guaranteed by construction because the action (16) is written in terms of differential forms. Thus, Noether's theorem provides a conserved current (A2) associated with this invariance [see the Appendix], given by

$$*J = -\Theta(\omega^{ab}, e^a, \delta \omega^{ab}) - I_{\xi} L, \quad (17)$$

where $\delta \omega^{ab} = -\mathcal{L}_{\xi} \omega^{ab}$, and the Lagrangian L can be read from Eq. (16). Then, Θ can be identified from Eq. (13) as

$$\Theta = -n \alpha_n \epsilon_{a_1 \dots a_{2n}} \mathcal{L}_{\xi} \omega^{a_1 a_2} \left[R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} + (-1)^n \frac{e^{a_3} \dots e^{a_{2n}}}{l^{2n-2}} \right]. \quad (18)$$

The useful identity $\mathcal{L}_{\xi} \omega^{ab} = DI_{\xi} \omega^{ab} + I_{\xi} R^{ab}$ allows writing the conserved current (17) as an exact form. Thus, the conserved charge can be written as

⁵Here we have defined the $2n$ -dimensional Euler density as $\mathcal{E}_{2n} = \epsilon_{a_1 \dots a_{2n}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}}$. Note that the normalization adopted here differs from standard mathematical convention as, for instance, in Ref. [14].

$$Q(\xi) = n\alpha_n \int_{\partial\Sigma} \epsilon_{a_1 \dots a_{2n}} I_\xi \omega^{a_1 a_2} \left[R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} + (-1)^n \frac{e^{a_3} \dots e^{a_{2n}}}{l^{2n-2}} \right]. \quad (19)$$

This expression can also be written as

$$Q(\xi) = \int_{\partial\Sigma} I_\xi \omega^{ab} \mathcal{T}_{ab}, \quad (20)$$

where \mathcal{T}_{ab} is the variation of the Lagrangian with respect Lorentz curvature

$$\mathcal{T}_{ab} = \frac{\delta L}{\delta R^{ab}}. \quad (21)$$

The general form adopted by the charge (20), can in fact be used for any suitable gravitational theory—possessing a unique cosmological constant—whose Lagrangian is a polynomial in the curvature R^{ab} and the vielbein e^a , and has the right boundary terms to ensure the action to have an extremum for ALAdS configurations.

It is noteworthy that this formula has been derived without making any assumptions about a background geometry. The ALAdS condition restricts only the local asymptotic relation between the curvature and the vielbein, with no mention of the global topology of the manifold.

If ξ is a Killing vector globally defined on the boundary $\partial\Sigma$, the surface integral (19) is the mass when $\xi = \partial_t$. Similarly, for other asymptotic Killing vectors, Eq. (19) gives finite values for the linear and angular momentum for a broad class of geometries. These statements are explicitly checked below for different ALAdS spacetimes with inequivalent topologies.

B. Examples

Schwarzschild-AdS black hole. The simplest example to be considered corresponds to the d -dimensional black hole solution for the EH action with cosmological constant, known as the Schwarzschild-AdS geometry

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2 d\Omega_{d-2}^2, \quad (22)$$

where $\Delta(r)^2 = 1 - 2M/r^{d-3} + r^2/l^2$. The only nonvanishing charge is associated with the timelike Killing vector ∂_t . Evaluating (19) on this metric yields

$$Q\left(\frac{\partial}{\partial t}\right) = M. \quad (23)$$

Kerr-AdS solution. In $d=2n$ dimensions, the rotating black hole solution is labeled by the mass and $n-1$ parameters which are related to the Casimir invariants of $\text{SO}(d-1)$. The one parameter Kerr-AdS spacetime, representing a

solution with mass and angular momentum along a single axis, is given by the following choice of vielbein [15]:

$$e^0 = \Delta_r [dt - a \sin^2(\theta) d\phi], \quad e^1 = \frac{1}{\Delta_r} dr,$$

$$e^3 = \sin(\theta) \Delta_\theta [adt - (r^2 + a^2) d\phi], \quad e^2 = \frac{1}{\Delta_\theta} d\theta,$$

$$e^i = r \cos(\theta) \tilde{e}^i, \quad (24)$$

where $i=5, \dots, d$, \tilde{e}^i is the vielbein for S^{d-4} and

$$\Delta_r^2 = \frac{(r^2 + a^2)(1 + r^2/l^2) - 2mr^{5-d}}{\Xi^2 \rho^2},$$

$$\Delta_\theta^2 = \frac{1 - a^2 \cos^2(\theta)/l^2}{\Xi^2 \rho^2},$$

$$\rho^2 = r^2 + a^2 \cos^2(\theta),$$

$$\Xi = 1 - a^2/l^2. \quad (25)$$

This geometry has two nonvanishing Noether charges, one associated with the timelike Killing vector ∂_t and the rotational Killing vector ∂_ϕ , respectively. For each dimension, the conserved charges depend on the parameters m, a . For instance, in six dimensions the mass and angular momentum are given by

$$Q\left(\frac{\partial}{\partial t}\right) = \frac{m}{\Xi^2}, \quad Q\left(\frac{\partial}{\partial \phi}\right) = \frac{ma}{2\Xi^2}. \quad (26)$$

(Un)wrapped brane solution. Unlike the Schwarzschild-AdS solution, where the spherical symmetry implies a manifest AdS asymptotic behavior—not only locally, but globally at the boundary—another kind of ALAdS geometry in d dimensions corresponds to a brane solution with flat transverse space,

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2(dx_1^2 + \dots + dx_{d-2}^2), \quad (27)$$

where $\Delta(r)^2 = -2m/r^{d-3} + r^2/l^2$. In this geometry at least one of the x^i coordinates must be compact, otherwise the parameter m can be absorbed by a coordinate transformation [16]. Assuming that the volume of the transverse space (x^i) is equal to V , the Noether charge associated with the Killing vector ∂_t is given by

$$Q\left(\frac{\partial}{\partial t}\right) = m \frac{V}{\Omega_{d-2}}, \quad (28)$$

and the corresponding charges related with spatial $\text{ISO}(d-2)$ symmetries are zero.

It should be noted that Eq. (28) depends on the topological nature of the transverse spatial section. In this case the

transverse space is compact, V is finite and so is the resulting Noether charge. When the transverse space is noncompact, the parameter m can be interpreted as a mass density.

This method provides the correct results even for the electrically charged extensions of the previous solutions. It is straightforward to prove that the formula works properly for the higher-dimensional Reissner-Nordström black hole, for the $(3+1)$ -dimensional Kerr-Newman solution, and for the electrically charged generalization of the (un)wrapped brane (27) studied in Ref. [17].

III. BORN-INFELD ACTION

In higher dimensions, besides the Einstein-Hilbert action one can consider other gravitational theories that include higher powers of the curvature and still yield second order field equations for the metric. Among them, there are a few that lead to well behaved ALAdS solutions [18]. In even dimensions ($d=2n$), the Born-Infeld (BI) action belongs to this class of theories. The BI action takes the explicit form

$$I = \frac{\kappa l^2}{2n} \int_{\mathcal{M}} \epsilon_{a_1 \dots a_d} \bar{R}^{a_1 a_2} \dots \bar{R}^{a_{d-1} a_d}, \quad (29)$$

where $\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b$. This theory is stationary for AL-AdS solutions and no boundary terms are required in order to have a well defined action principle.⁶

Following the Hamiltonian method, it would be extremely difficult to obtain a mass formula as a surface integral for an arbitrary localized matter distribution in this kind of theories. Such construction would require inverting the symplectic matrix of this action. However, the rank of this matrix depends on the fields and therefore no general form can be found for an arbitrary field configuration. On the other hand, in the Lagrangian formalism, the Noether current for diffeomorphisms is an exact form, which allows writing down the conserved charge at once as the surface integral

$$Q(\xi) = \int_{\partial\Sigma} I_\xi \omega^{ab} \mathcal{T}_{ab}, \quad (30)$$

where

$$\mathcal{T}_{ab} = \frac{\kappa l^2}{2} \epsilon_{aba_3 \dots a_d} \bar{R}^{a_3 a_4} \dots \bar{R}^{a_{d-1} a_d}. \quad (31)$$

This is an appropriate definition of mass and other conserved charges, as is shown in the following examples.

Static spherically symmetric solution. The spherically symmetric black hole solution of BI action was studied in

⁶This expression can also be written as $L = 2^{n-1}(n-1)! \kappa l^2 \sqrt{\det(R^{ab} + l^{-2} e^a e^b)}$, and for this reason it has been dubbed the Born-Infeld action [13]. In four dimensions, this action reduces to the usual Einstein-Hilbert with cosmological constant plus the Euler density, with all coefficient fixed as in previous section.

Ref. [19]. In $3+1$ dimensions this is the Schwarzschild-AdS geometry, but differs from it in higher dimensions. The line element is given by

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2 d\Omega_{d-2}^2, \quad (32)$$

with $\Delta(r)^2 = 1 - (2M/r)^{1/(n-1)} + r^2/l^2$.

The mass is obtained by direct computation of Eq. (30) for the Killing vector $\xi = \partial_t$,

$$Q\left(\frac{\partial}{\partial t}\right) = M. \quad (33)$$

The conserved charges associated with rotational isometries vanish.

(Un)wrapped brane solution. A feature in common for the BI and EH actions is possessing a set of solutions that are only ALAdS, but not globally AdS at the boundary. Among many solutions, it is interesting to consider the analog of the (un)wrapped brane (27), for which the line element reads

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2(dx_1^2 + \dots + dx_{d-2}^2), \quad (34)$$

with $\Delta(r)^2 = -(2m/r)^{1/(n-1)} + r^2/l^2$. This corresponds to a particular case of the class of geometries studied by Cai and Soh in Ref. [20]. The transverse space in this gravitational configuration is (locally) flat, with a volume V . This geometry has just one nonvanishing Noether charge, that is the density of mass associated with the Killing vector ∂_t

$$Q\left(\frac{\partial}{\partial t}\right) = m \frac{V}{\Omega^{d-2}}. \quad (35)$$

This last result is in complete agreement with the one computed by the Hamiltonian method, using a minisuper-space model applied to configurations with transverse space not necessarily compact. However, this result differs by a global factor compared to the same case as treated in Ref. [20]. The origin of this mismatch lies in the fact that transverse space is no longer spherically symmetric; therefore, the volume V cannot cancel the normalization factor, fixed beforehand to give the correct value of mass for spherically symmetric black holes.

IV. CONSERVED CHARGE FOR LORENTZ TRANSFORMATIONS

Apart from charges associated with diffeomorphisms and due to the invariance of the EH and BI actions under local Lorentz transformations, the Noether method can also be applied to obtain conserved quantities for these symmetries (see the Appendix). Substituting $\delta\omega^{ab} = -D\lambda^{ab}$ in the general expression for the Noether current (A2)

$$*J = \delta\omega^{ab} \mathcal{T}_{ab}, \quad (36)$$

where \mathcal{T}_{ab} is covariantly constant, yields the conserved charge in terms of the parameter of the Lorentz transformation λ^{ab} as

$$Q(\lambda^{ab}) = \int_{\partial\Sigma} \lambda^{ab} \mathcal{T}_{ab}. \quad (37)$$

Here

$$\mathcal{T}_{ab} = n \alpha_n \epsilon_{aba_3 \dots a_{2n}} \left[R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} + (-1)^n \frac{e^{a_3} \dots e^{a_{2n}}}{l^{2n-2}} \right], \quad (38)$$

for the Einstein-Hilbert case and

$$\mathcal{T}_{ab} = \frac{\kappa l^2}{2} \epsilon_{aba_3 \dots a_d} \bar{R}^{a_3 a_4} \dots \bar{R}^{a_{d-1} a_d}, \quad (39)$$

for the Born-Infeld action.

Lorentz covariance. The formula (37) is a scalar from the point of view of Lorentz covariance. On the other hand, the charges (30) and (19) associated with diffeomorphism invariance transform under local Lorentz rotations as

$$\delta_\lambda Q(\xi) = - \int_{\partial\Sigma} \mathcal{L}_\xi \lambda^{ab} \mathcal{T}_{ab}. \quad (40)$$

This change in $Q(\xi)$ vanishes under the usual assumption that the local transformation with parameter λ^{ab} approaches a rigid Lorentz transformation on $\partial\Sigma$, constant along ξ , that is, $\mathcal{L}_\xi \lambda^{ab}|_{\partial\Sigma} = 0$.

V. SUMMARY AND PROSPECTS

The method presented here is the direct application of Noether's theorem to a first order gravitational action $I[e, \omega]$, provided the spacetime satisfies ALAdS boundary conditions. The analysis leads directly to general analytic expressions for the conserved charges, both for the Einstein-Hilbert and Born-Infeld actions. The treatment is entirely Lagrangian and yields values for the charges that match exactly those obtained by Hamiltonian methods (e.g., ADM). In the Hamiltonian approach, however, when the space is not asymptotically flat it is often necessary to renormalize the asymptotic Killing vectors to define the conserved charges (see, for example, Ref. [24]).

The resulting charges are finite for localized distributions of matter (black holes) and yield finite density formulas for extended objects (e.g., strings). There is no need to subtract the ‘‘vacuum’’ energy in order to regularize the charges. It could be argued that the Euler density added as a boundary term does this job for us, but what is indisputable is the fact that one does not need to specify a reference background against which one should compute the value of the charges. What could be even more surprising is the fact that the formulas (19), for EH action, and Eq. (30) for BI the case, give

the correct answers even for radically different asymptotic behaviors.

The general nature of the treatment, allows extending to $2n$ dimensions, and also to the BI action, the following result valid for the EH action in 3+1 dimensions with a negative cosmological constant: Noether charges associated to Lorentz and diffeomorphism invariance vanish identically in locally AdS spacetimes. As can be directly checked from Eqs. (19) and (37), the charges are identically zero if

$$\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b = 0$$

in the bulk. This means that spaces which are locally AdS have vanishing charges. In particular, any locally AdS geometry with a timelike Killing vector should have zero mass.⁷ This brings in an interesting issue: there could be several topologically different spaces with locally AdS geometry for which all their quantum numbers associated to spatial transformations vanish. Each of these spaces could be reasonably used as vacuum for a quantum field theory and one should also expect to find interpolating instanton or soliton configurations. Also, any massive solution such as the examples discussed above could be seen as an excitation of the corresponding background in the same topological sector.

Prospects. Only two cases (EH and BI) have been considered here among all the possible Lanczos-Lovelock theories of gravity [21–23]. The suitable theories describing gravitation in higher dimensions must possess a unique cosmological constant and therefore, a unique background in each topological sector, so that vacuum configurations approach to local AdS spacetimes with a fixed curvature radius at the boundary [18]. Indeed, there exists a subset of these theories which possesses well behaved black hole solutions [26]. The extension of this formalism to those theories, in even dimensions, will be discussed elsewhere.

As mentioned above, the odd-dimensional manifolds cannot be treated with the same method presented here. The cases of interest in $(2n+1)$ dimensions, analogous to those discussed in this paper, would be EH and Chern-Simons. Regularization of the charges in these cases remains an open problem in the presented framework.

Another interesting problem to address is the classification of all $2n$ -dimensional constant curvature spaces, as they can be thought of as candidates for vacuum configurations for an AdS field theory. Certainly, one possible class of such spaces could be AdS with identifications along global Killing vectors (that do not introduce causal or conical singularities), but it is not obvious that this exhausts all possibilities in high enough dimensions.

⁷It should be stressed that this assertion is only valid in even dimensions, for it is well known that at least in three dimensions different locally AdS spaces can have different energies ($m = -1$ for AdS, $m \geq 0$ for black holes [25]). This probably means that the analysis presented here cannot be repeated *verbatim* in odd dimensions.

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APPENDIX: NOETHER THEOREM

In order to fix the notation and conventions, here we briefly review Noether’s theorem. Consider a d -form La-

grangian $L(\varphi, d\varphi)$, where φ denotes collectively a set of p -form fields. An arbitrary variation of the action under a local change $\delta\varphi$ is given by the integral of

$$\delta L = (\text{EOM}) \delta\varphi + d\Theta(\varphi, \delta\varphi), \tag{A1}$$

where EOM stands for equations of motion and Θ is a corresponding boundary term [27]. The total change in φ [$\bar{\delta}\varphi = \varphi'(x') - \varphi(x)$] can be decomposed as a sum of a local variation and the change induced by a diffeomorphism, that is, $\bar{\delta}\varphi = \delta\varphi + \mathcal{L}_\xi\varphi$, where \mathcal{L}_ξ is the Lie derivative operator. In particular, a symmetry transformation acts on the coordinates of the manifold as $\delta x^\mu = \xi^\mu(x)$, and on the fields as $\delta\varphi$, leading a change in the Lagrangian given by $\delta L = d\Omega$.

Noether’s theorem states that there exists a conserved current given by

$$*J = \Omega - \Theta(\varphi, \delta\varphi) - I_\xi L, \tag{A2}$$

which satisfies $d*J = 0$. This, in turn, implies the existence of the conserved charge

$$Q = \int_\Sigma *J,$$

where Σ is the spatial section of the manifold, when a manifold is assumed to be of topology $R \times \Sigma$.

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