

$\mathcal{N} = 1$ SUGRA Noether Charges

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In this work a generic set of boundary conditions for $\mathcal{N} = 1$ SUGRA is proposed. This conditions defines that Hamiltonian charges equals Noether ones, including supercharge.

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I. INTRODUCTION

The role of boundary conditions in any theory of fields is manifold and fundamental. First the boundary conditions are necessary to solve the equations of motion. They also are necessary to have a proper action principle. Furthermore *they define the ensemble in which the theory exists* (for gravity see for instance [1]).

In general the variation of the action, as a functional of a field $\phi(x)$, is given by

$$\delta I = \int_{\mathcal{M}} EM \delta \phi + d\Theta(\phi, \delta \phi),$$

where EM stands for the equations of motion. Now the action is called stationary whether the boundary term $\Theta(\phi, \delta \phi)$ vanishes on shell, which only happens provided adequate boundary conditions. Only in this case one has a proper action principle.

For gravity the issue of an adequate set of boundary conditions has been certainly addressed by many authors, in many different problems and approaches ([2, 3, 4, 5, 6], etc. See also [7]). In [7, 8] for locally asymptotical AdS spaces was introduced a new set of boundary condition at the asymptotical space region. As a result of these new boundary conditions Noether and Hamiltonian charges for diffeomorphisms agree. The intension of this work was extend the results in [7, 8] to four dimensional $\mathcal{N} = 1$ supergravity (SUGRA). The results obtained are analogous.

To simplify the calculations only $\mathcal{N} = 1$ SUGRA will be discussed, however most the computations can be readily extended to $\mathcal{N} > 1$ SUGRA's. This particular supergravity in four dimensions is a simpler laboratory to test technics to later be applied in higher dimensions or $\mathcal{N} > 1$. In eleven dimensions, for instance, although there is no standard SUGRA with a negative cosmological constant [9] there is the eleven dimension Chern Simons SUGRA [10].

The space to be discussed in this work is given by $\mathcal{M} = \mathbb{R} \times \Sigma$ where Σ corresponds to a 3-dimensional spacelike hypersurface and \mathbb{R} stands for the time direction. The space possesses an asymptotical locally AdS region, which defines the boundary $\mathbb{R} \times \partial\Sigma_{\infty}$.

For simplicity in this work the differential forms language will be used, and \wedge product between differential forms will be omitted.

II. SUPERGRAVITY

The four dimensional $\mathcal{N} = 1$ supergravity action with negative cosmological constant is defined by the Lagrangian [11]

$$\mathbf{L} = \frac{l^2}{64\pi} \bar{R}^{ab} \bar{R}^{cd} \epsilon_{abcd} + \bar{\Psi} \gamma_5 \gamma_a e^a D\Psi, \quad (1)$$

where $D = d + \frac{1}{4} \omega^{ab} \gamma_{ab} + \frac{1}{2l} e^a \gamma_a$ and $\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b$. Here e^a is the vierbein and ω^{ab} the spin connection, R^{ab} is the two form of curvature defined as

$$R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} = \frac{1}{2} R^{ab}_{cd} e^c e^d$$

where R^{ab}_{cd} is the Riemann Tensor. Ψ stands for the Ravita Schwinger field.

From the Lagrangian (1) one obtains the equations of motion

$$\frac{l^2}{32\pi} \bar{R}^{ab} e^c \epsilon_{abcd} + \bar{\Psi} \gamma_5 \gamma_d D\Psi - \frac{1}{2} \bar{\Psi} \gamma_5 \gamma_a e^a \gamma_d \Psi = 0, \quad (2a)$$

$$\frac{l^2}{32\pi} D(\bar{R}^{cd} \epsilon_{abcd}) + \frac{1}{2} \bar{\Psi} \gamma_5 \gamma_c e^c \gamma_{ab} \Psi = 0, \quad (2b)$$

$$\gamma_5 \gamma_c e^c D\Psi = 0, \quad (2c)$$

$$D(\bar{\Psi} \gamma_5 \gamma_c e^c) = 0. \quad (2d)$$

and the boundary term

$$\Theta = \frac{l^2}{32\pi} (\delta_0 \omega^{ab} \bar{R}^{cd} \epsilon_{abcd}) + \bar{\Psi} \gamma_5 \gamma_a e^a \delta_0 \Psi. \quad (3)$$

Remarkably Eq.(2b) can be rewritten in terms of the torsion two form $T^a = de^a + \omega^a_b e^b$ simply as

$$T^a = -4\pi \bar{\Psi} \gamma^a \Psi. \quad (4)$$

The torsion two form contains the torsion tensor, T^a_{bc} , as $T^a = \frac{1}{2} T^a_{bc} e^b e^c$.

III. BOUNDARY CONDITIONS

In order to solve the equations of motion (2) is obviously necessary to provide boundary conditions. In Ref.[7] a new set boundary conditions was introduced for the bosonic sector of action (1), namely the ALAdS

condition. This boundary conditions is based on that for any asymptotically locally AdS space one can define a pseudo connection $W = 1/2 W^{AB} J_{AB}$, with

$$W^{AB} = \begin{bmatrix} \omega^{ab} & l^{-1}e^a \\ -l^{-1}e^a & 0 \end{bmatrix},$$

whose field strength $F = 1/2 F^{AB} J_{AB} = dA + A \wedge A$, with

$$F^{AB} = \begin{bmatrix} \bar{R}^{ab} & l^{-1}T^a \\ -l^{-1}T^a & 0 \end{bmatrix},$$

vanishes on the asymptotic spatial region. Since for the standard solutions $T^a = 0$ everywhere, the condition restricts only the curvature \bar{R}^{ab} .

To incorporate the fermionic sector one option is to follow the same underlying idea, namely that the (super)-field strength vanishes at asymptotical region. Analogously it is defined a (super)-connection $A = 1/2 \omega^{ab} J_{ab} + l^{-1}e^a J_a + \bar{\Psi}Q$ whose field strength reads

$$F = \frac{1}{2} \left(\bar{R}^{ab} + \frac{1}{2} \bar{\Psi} \gamma^{ab} \Psi \right) J_{ab} + \left(\frac{T^a}{l} + \frac{1}{2} \bar{\Psi} \gamma^a \Psi \right) J_{a5} + D\bar{\Psi}Q. \quad (5)$$

It is direct to see that to impose the vanishing of the field strength (5) at the asymptotical region in turn implies that asymptotically $\bar{R}^{ab} \rightarrow -\frac{1}{2} \bar{\Psi} \gamma^{ab} \Psi$, $T^a \rightarrow -\frac{1}{2} \bar{\Psi} \gamma^a \Psi$ and $D\bar{\Psi} \rightarrow 0$. Unfortunately these boundary conditions do not imply the vanishing of the boundary term (3). To overcome this one can also impose that Ψ vanish at the asymptotical region. This condition was determined in [4] in another approach (See also [2]). From the point of view of the gauge super group Ψ is part of the connection, therefore to impose $\Psi = 0$ at the asymptotical region can be regarded as a gauge fixing.

The vanishing of Ψ together with the vanishing of F imply that at the asymptotic region

$$\bar{R}^{ab} \rightarrow 0 \text{ and } T^a \rightarrow 0,$$

namely that \mathcal{M} is actually an asymptotically locally anti de Sitter space.

Now it is straightforward to prove that under these boundary conditions just proposed, *i.e.*,

$$F(x) \rightarrow 0 \text{ as } x \rightarrow \mathbb{R} \times \partial\Sigma_\infty,$$

the boundary term (3) vanishes for arbitrary variations of the fields.

IV. DIFFEOMORPHISMS NOETHER CURRENT

The invariance under diffeomorphisms of the Lagrangian (1) yields a Noether current (See (A3) in

appendix) which, after a straightforward computation, reads

$$*\mathbf{J}_\xi = -d \left(\frac{1}{32\pi} I_\xi \omega^{ab} \bar{R}^{cd} \epsilon_{abcd} + \bar{\Psi} \gamma_5 \gamma_a e^a I_\xi \Psi \right). \quad (6)$$

To define a conserved charge from the current in Eq.(6) one must restrict ξ to be a global isometry, namely a Killing vector. On the other hand, since Eq. (6) is an exact form the conserved charge to be obtained form it, through integration, depends only on its asymptotical value.

Note that the boundary condition at the asymptotical spatial region requires that Ψ vanish yielding the vanishing of the second term in Eq. (6). The same result can be obtained smoothly by proposing for the gravitino a fall-off behavior as $\Psi \sim r^{-3/2}$ (See Ref. [4]). This implies that any conserved charge associated with the global diffeomorphisms reads

$$Q_\xi = -\frac{1}{32\pi} \int_{\Sigma_\infty} I_\xi \omega^{ab} \bar{R}^{cd} \epsilon_{abcd}.$$

Remarkably this result is the same obtained in Ref. [7] for only the bosonic part of Eq.(1). This result indicates that the value of mass or angular momenta as the asymptotical value of an integral should encode the presence of the gravitinos. This not usual, for instance, in the case of Reissner Nordström solution the expression of the charge associated time symmetry reads

$$Q \left(\frac{\partial}{\partial t} \right) \sim M - \frac{e^2}{r},$$

which obviously only equals M , the mass, as $r \rightarrow \infty$.

This result also might let room for the detection of gravitinos as a correction to the value of the mass at short distance.

V. SUPERSYMMETRIC NOETHER CURRENT

It is well known that the Lagrangian (1) is (on-shell) also invariant under the supersymmetry transformation

$$\begin{aligned} \delta_\epsilon e^a &= 8\pi \bar{\epsilon} \gamma^a \Psi, \\ \delta_\epsilon \Psi &= D\epsilon, \\ \delta_\epsilon \omega^{ab} &= \pm \frac{1}{2} \epsilon^{abcd} \bar{\epsilon} \gamma_5 \gamma_c E_d^\nu (D(\Psi))_{\mu\nu}, \end{aligned} \quad (7)$$

where ϵ is a Grassmann valued 0-form and E_a^μ is the inverse of e_μ^a .

After some manipulations it can be shown the Lagrangian (1) changes under the supersymmetric transformation (7) in the boundary term

$$\alpha = \frac{l^2}{32\pi} (\delta_\epsilon \omega^{ab} \bar{R}^{cd} \epsilon_{abcd}) + \bar{\epsilon} \gamma_5 \gamma_a e^a D\Psi,$$

and thus this supersymmetric transformation give rise to the Noether current $*\mathbf{J}_\epsilon = \Theta - \alpha$ which reads

$$*\mathbf{J}_\epsilon = -d(\bar{\epsilon} \gamma_5 \gamma_a e^a \Psi). \quad (8)$$

Similarly one can define a charge associated with Eq.(8) by integrating at the asymptotical spatial region as

$$Q_\epsilon = - \int_{\partial\Sigma_\infty} (\bar{\epsilon}\gamma_5\gamma_a e^a\Psi). \quad (9)$$

Remarkably, after some algebraic manipulations, Eq. (9) reproduces the Hamiltonian prescription for the super charge obtained in [12].

VI. HAMILTONIAN VERSUS NOETHERIAN

As far it has been found expressions for the Noether charges associated diffeomorphisms and supersymmetry. However charges as mass and angular momenta or supercharges necessarily are defined within the context of Hamiltonian formalism. To establish a connection between Noether and Hamiltonian charges the covariant phase space method develop in [13] can be used. For diffeomorphisms the variation of the Hamiltonian Charge G_ξ reads

$$\begin{aligned} \delta G_\xi = & \int_{\partial\Sigma} \delta \left(\frac{1}{32\pi} I_\xi \omega^{ab} \bar{R}^{cd} \epsilon_{abcd} + \bar{\Psi} \gamma_5 \gamma_a e^a I_\xi \Psi \right) \\ & + I_\xi \Theta(\delta\omega, \delta\Psi, \omega, \Psi, e), \end{aligned} \quad (10)$$

where $\Theta(\delta\omega, \delta\Psi, \omega, \Psi, e)$ is given by Eq.(3). However the second term in Eq.(10) was tailored to vanish in the asymptotical spatial region for arbitrary variations of the fields, thus

$$\delta G_\xi = \delta Q_\xi. \quad (11)$$

In this way the result obtained in Ref. [8] for the bosonic part of Eq. (1) extends to SUGRA with $\mathcal{N} = 1$. Furthermore note that this result allows to conjecture, given the structure of Eq.(10), that this result could be extended to others SUGRA with negative cosmological constant in higher dimensions and $\mathcal{N} \neq 1$.

Similarly the expression of the Noether charge associated with supersymmetry can be proven to be equivalent to the Hamiltonian one, *i.e.*

$$\delta G_\epsilon = \delta Q_\epsilon. \quad (12)$$

This results is not surprising since Eq.(9) corresponds to the Hamiltonian prescription.

VII. CONCLUSIONS

In this work were analyzed the Noether currents associated with symmetry of diffeomorphisms and the supersymmetry of the $\mathcal{N} = 1$ supergravity in four dimensions with a negative cosmological constant. It was proven that it is possible to define a proper set of boundary conditions which not only defines a sensible action principle,

but also determines that Noether charges equal Hamiltonian ones. It is worth to mention that the results obtained in this work are equivalent to those in [5] in terms of superpotentials.

The boundary conditions proposed in this paper are not new but corresponds to a combination of an extension of previous results of the bosonic part of the action and previously known prescription for the asymptotical behavior of gravitinos.

The generic form in which the results arise allows to conjecture that the basic results of this work should extend to other SUGRA in higher dimensions.

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APPENDIX A: NOETHER METHOD

Any infinitesimal transformation of a field $\phi(x)$ can be split as a local functional transformation plus a diffeomorphism, given by $x' = x + \xi(x)$ (See [14]). This can be done as follows,

$$\delta\phi(x) = \phi'(x') - \phi(x) = \phi'(x') - \phi'(x) + \phi'(x) - \phi(x) \quad (A1)$$

where $\phi'(x) - \phi(x) = \delta_0\phi(x)$ is a local functional transformation, and $\phi'(x') - \phi'(x) = \mathcal{L}_\xi\phi$ corresponds to the Lie derivative along the vector field ξ , thus $\delta\phi = \delta_0\phi + \mathcal{L}_\xi\phi$. For any Lagrangian this yields

$$\delta\mathbf{L} = (E.M.)\delta_0\phi + d\Theta(\phi, \delta_0\phi) + dI_\xi\mathbf{L}. \quad (A2)$$

A symmetry is defined as a change in the field configuration that does not alter the field equations. To satisfy that $\delta\phi$ has to be such that the Lagrangian changes in a total derivative, *i.e.*, $\delta\mathbf{L} = d\alpha$, thus on shell the current

$$*\mathbf{J}_\xi = \Theta(\delta_0\phi, \phi) + I_\xi\mathbf{L} - \alpha, \quad (A3)$$

satisfies $d(*\mathbf{J}_\xi) = 0$, which is usually called the Noether current.

APPENDIX B: DEFINING THE ADS ALGEBRA

In this work is used the algebra definition

$$\begin{aligned} [J_{AB}, J_{CD}] &= -\delta_{AB}^{EF} \delta_{CD}^{GH} \eta_{EG} J_{FH} \\ [Q^\alpha, J_{CD}] &= (\gamma_{CD})^\alpha_\beta Q^\beta \\ \{Q^\alpha, Q^\beta\} &= \frac{1}{2} (\gamma^{AB} C^{-1})^{\alpha\beta} J_{AB} \end{aligned}$$

Additionally since the Clifford algebra is realized by the γ matrices as $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, the AdS generators have the representation given by

$$J_{ab} = \frac{1}{2} \gamma_{ab} \text{ and } J_{a5} = \frac{1}{2} \gamma_a$$

with $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$.

In addition the representation satisfies that,

$$\begin{aligned} C^T &= -C \\ C\gamma_a C^{-1} &= -\gamma_a^T. \end{aligned}$$

APPENDIX C: USEFUL RELATIONS

During the calculations in this work the useful identity of the commutator the covariant derivative $D = d + \frac{1}{4}\omega^{ab}\gamma_{ab} + \frac{1}{2l}e^a\gamma_a$ and the lie derivative, which reads,

$$[\mathcal{L}_\xi, D]\Psi = (\mathcal{L}_\xi\omega^{ab})J_{ab}\Psi = (D(I_\xi\omega^{ab}) + I_\xi R^{ab})\Psi, \quad (\text{C1})$$

was necessary.

In addition these other identities

$$\begin{aligned} \gamma_5 &= \gamma_0\gamma_1\gamma_2\gamma_3 \\ \epsilon^{abcd}\gamma_{cd} &= 2\gamma_5\gamma^{ab} \\ \gamma_{ab}\gamma_c &= (\eta_{bc}\gamma_a - \eta_{ac}\gamma_b) + \epsilon_{abcd}\gamma_5\gamma^d \\ \gamma_c\gamma_{ab} &= (\eta_{ca}\gamma_b - \eta_{ab}\gamma_a) + \epsilon_{abcd}\gamma_5\gamma^d \\ \gamma_a\gamma_{cd}\gamma^a &= 0 \\ \gamma_{cd}\gamma_a\gamma^{cd} &= 0 \end{aligned}$$

for the gamma matrices were necessary as well.

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